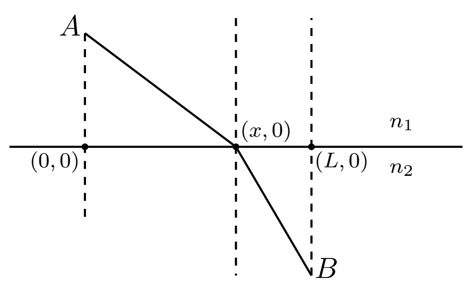
PHY 71100: ANALYTICAL DYNAMICS

Problem Set 1

Due September 16, 2024

Problem 1 (6 points)

Consider two optically different materials with indices of refraction n_1 and n_2. There is a planar interface between the two as shown. A ray of light propagates from point $A = (0, y_1)$ in the first medium to point $B = (L, -y_2)$ in the second, striking a point $(x, 0)$ on the interface. Obtain the time of propagation from A to B, keeping in mind that speed of propagation of light is c/n in a medium of refractive index n. Minimize your expression for the total time as a function of x and show that this leads to Snell's law. (This is Fermat's principle of least time which applies to light propagation. This was part of the inspiration for formulating the variational principle for mechanics.)



Solution

The distance from A to the point of incidence $(x, 0)$ is

$$d_1 = \sqrt{(0-x)^2 + (y_1 - 0)^2} = \sqrt{x^2 + y_1^2}$$

The time of travel from A to $(x, 0)$ is thus

$$T_1 = \frac{n_1}{c}\sqrt{x^2 + y_1^2}$$

Likewise, the time of travel from $(x, 0)$ to B is given by

$$T_2 = \frac{n_2}{c}\sqrt{(L-x)^2 + (-y_2)^2}$$

To extremize the time of travel, we must choose x such that $\partial/\partial x$ of the total time $T_1 + T_2$ is zero; i.e.,

$$\frac{\partial}{\partial x}(T_1 + T_2) = 0$$

Explicitly, this becomes

$$n_1 \frac{x}{\sqrt{x^2 + y_1^2}} = n_2 \frac{L - x}{\sqrt{(L - x)^2 + y_2^2}}$$

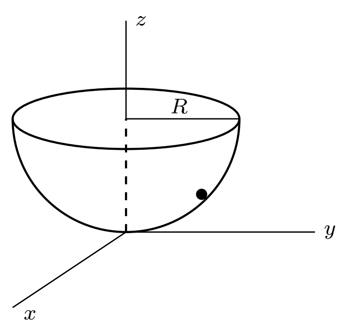
If we denote the angle of the light ray with the vertical y-axis as θ, this equations can be written as

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

This is Snell's law. This can be interpreted as saying that the trajectory of light between two points is given by extremizing the time of travel between the same two points.

Problem 2 (10 points)

A particle can slide without friction on the inner surface of a hemispherical bowl (of negligible thickness) which is resting on the ground as shown. The radius of the bowl is R. Obtain the Lagrangian and equations of motion of the particle. (You should keep in mind that the particle can have angular motion as well as radial motion. The particle cannot get off the surface, so that the vertical motion is related to the radial motion.)



Solution

Cylindrical coordinates are the simplest for this problem. Consider the horizontal plane which contains the particle; this is at height z and the circle obtained by this plane intersecting the bowl has radius r. Then we have

$$(R - z)^2 + r^2 = R^2$$

Let $R - z = \xi$. This equation reads $\xi^2 + r^2 = R^2$ and, upon differentiation, we also get $\dot{r} = -\xi\dot{\xi}/r$. We use the plane polar coordinates for the (x, y)-plane. Thus $x = r\cos\varphi$, $y = r\sin\varphi$. The kinetic term is then

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2) = \tfrac{1}{2}m\left((1 + \frac{\xi^2}{r^2})\dot{\xi}^2 + r^2\dot{\varphi}^2\right) = \tfrac{1}{2}m\frac{R^2}{R^2 - \xi^2}\dot{\xi}^2 + \tfrac{1}{2}m(R^2 - \xi^2)\dot{\varphi}^2$$

The potential energy is $V = mgz = mgR - mg\xi$.

$$L = m\left[\tfrac{1}{2}\frac{R^2}{R^2-\xi^2}\dot{\xi}^2 + \tfrac{1}{2}(R^2-\xi^2)\dot{\varphi}^2 + g\xi\right] + \text{constant}$$

We then get

$$\begin{aligned} \frac{\partial L}{\partial \dot{\xi}} &= m\frac{R^2}{R^2-\xi^2}\dot{\xi}, & \frac{\partial L}{\partial \dot{\varphi}} = m(R^2-\xi^2)\,\dot{\varphi} \\ \frac{\partial L}{\partial \xi} &= m\left[\frac{R^2\xi\dot{\xi}^2}{(R^2-\xi^2)^2} - \xi\dot{\varphi}^2 + g\right], & \frac{\partial L}{\partial \varphi} = 0 \end{aligned}$$

These give the equations of motion

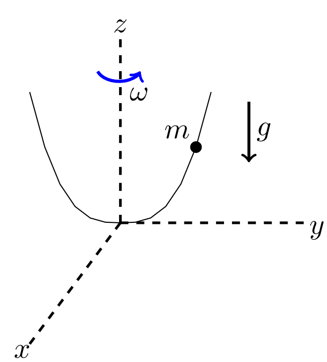
$$\begin{aligned} \frac{R^2\ddot{\xi}}{R^2-\xi^2} + \frac{R^2\,\xi\dot{\xi}^2}{(R^2-\xi^2)^2} &= g - \xi\,\dot{\varphi}^2 \\ \frac{d}{dt}\left[(R^2-\xi^2)\dot{\varphi}\right] &= 0 \end{aligned}$$

Problem 3 (10 points)

A thin wire is bent in the shape of a curve given by $z = k\,r^3/3$, where $r^2 = x^2 + y^2$. It is placed on the ground as shown and spun around the vertical axis (z-axis) with a constant angular velocity ω. A bead of mass m can slide frictionlessly along the wire.

a) Obtain the Lagrangian and the equations of motion

b) Show that there are two equilibrium points where the force on the particle vanishes.



Solution

a) In this case, it is again simpler to use cylindrical coordinates. The position of the bead can be written as

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad z = \frac{kr^3}{3}$$

Since the bead stays on the wire, the only angular motion it has is due to the rotation, so we can take $\varphi = \omega t$, where $\omega = \dot{\varphi}$ is the angular velocity. Thus we find

$$\dot{x} = \dot{r}\cos\varphi - \omega r\sin\varphi, \qquad \dot{y} = \dot{r}\sin\varphi + \omega r\cos\varphi, \qquad \dot{z} = kr^2\dot{r}$$

The kinetic energy is thus given by

$$T = \frac{m}{2}\left[\dot{r}^2 + \omega^2 r^2 + k^2 r^4 \dot{r}^2\right]$$

The potential energy is $V = mgz = mgkr^3/3$, so that the Lagrangian is

$$L = \frac{m}{2}\left[(1 + k^2 r^4)\dot{r}^2 + \omega^2 r^2\right] - m\frac{gkr^3}{3}$$

There is only one dynamical variable r, since z is determined by $z = \frac{kr^3}{3}$. The relevant derivatives are

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}(1 + k^2 r^4), \qquad \frac{\partial L}{\partial r} = m\left[2k^2 r^3 \dot{r}^2 + \omega^2 r - gkr^2\right]$$

The equation of motion is thus given as

$$(1 + k^2 r^4)\ddot{r} + 2k^2 r^3 \dot{r}^2 - \omega^2 r + gkr^2 = 0$$

b) From the equation of motion, we see that the particle will have zero acceleration, i.e., $\ddot{r} = 0$, if $\dot{r} = 0$ and we place the particle where $gkr^2 - \omega^2 r = 0$. These would be the equilibrium points. Evidently, there are two, since this is a quadratic equation. The points are given by

$$r_1 = 0, \qquad r_2 = \frac{\omega^2}{gk}$$