PHY 71100: ANALYTICAL DYNAMICS

Problem Set 2

Due September 25, 2024

Problem 1

Consider the motion of a point-particle of mass m in an attractive central potential $V(r) = kr^2/2$, where k is a real positive constant. The motion can be considered to be in a plane and the Lagrangian is given by

$$\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r)$$

Show that all orbits are bounded and that there is a minimum energy $E_{\rm min} = (l^2k/m)^{\frac{1}{2}}$, where l is the angular momentum.

Solution

In this case, we again have a central potential, so $mr^2\dot{\varphi} = l$. The effective potential is

$$U_{\rm eff} = \frac{l^2}{2mr^2} + V(r)$$

The energy is given by

$$E = \tfrac{1}{2}m\dot{r}^2 + U_{\rm eff}$$

Since $\dot{r}^2 \geq 0$, we must have $E \geq U_{\rm eff}$. Thus E must be greater than the minimum value of $U_{\rm eff}$. From the formula for $U_{\rm eff}$, the minimum occurs at

$$\frac{\partial U_{\rm eff}}{\partial r} = 0$$

For $V = \frac{1}{2}kr^2$, this gives

$$-\frac{l^2}{mr^3} + kr = 0 \qquad \Longrightarrow \qquad r^2 = \sqrt{\frac{l^2}{mk}}$$

The value of $U_{\rm eff}$ at this point (i.e., the minimum value of $U_{\rm eff}$) is

$$U_{\rm eff}|_{\rm min} = \sqrt{\frac{l^2k}{m}} \qquad \Longrightarrow \qquad E \geq \sqrt{\frac{l^2k}{m}}$$

The turning points of the motion in an orbit have $\dot{r} = 0$, so the corresponding values of r are given by the equation $E = U_{\rm eff}(r)$. To see that all orbits are bounded, notice that $U_{\rm eff}$ is positive and goes to ∞ as $r \to 0, \infty$. Thus, for any finite positive value of E (E cannot be negative, since $E > \sqrt{l^2k/m}$), we have two solutions to the condition $E = U_{\rm eff}(r)$, corresponding to $r_{\rm min}$ and $r_{\rm max}$. In fact these values are given by

$$r^2 = \frac{E}{k} \pm \frac{1}{k}\sqrt{E^2 - (l^2k/m)}, \qquad + \text{ for } r_{\rm min}, \; - \text{for } r_{\rm max}$$

These values are bounded and hence every orbit is bounded.

Problem 2

A particle of mass m can move in two dimensions under the influence of a central force

$$F = -\frac{\partial V}{\partial r} = -\frac{ar}{b}\exp\left(-\frac{r^2}{2b}\right)$$

where r is the radial distance and a, b are positive constants.

a) Find the potential energy $V(r)$ corresponding to this force.

b) For what range of values of the energy is the motion bounded?

c) If an orbit is circular, what should be the relation between the angular momentum and the radius of the orbit? (You will get a transcendental equation, you do not have to solve it; just give this relation.)

Solution

a) The potential energy V is given by

$$F = -\frac{\partial V}{\partial r} \implies V = -\int F dr$$

Since $\partial_r(e^{-r^2/2b}) = -(r/b)e^{-r^2/2b}$, we get

$$V(r) = -a\ \exp\left(-\frac{r^2}{2b}\right)$$

There is a possible additive constant, but we set this to zero choosing $V(\infty) = 0$. (For other choices, we will have a constant in the Lagrangian, but it will not affect the equations of motion.)

b) The effective potential is given by

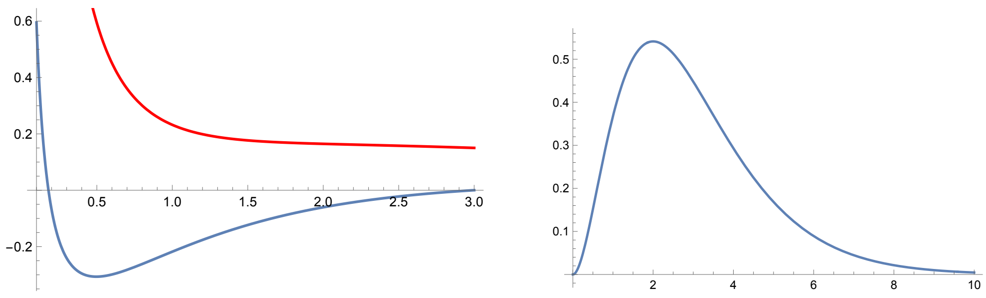
$$U_{\rm eff} = \frac{l^2}{2mr^2} - a\, e^{-r^2/2b}$$

This is positive and large as $r \to 0$, and also positive for large r since the second term dies out exponentially. For bound orbits, there should be $r_{\rm min}$ and $r_{\rm max}$ at which $E = U_{\rm eff}$. This means that we should have a region of $U_{\rm eff}$ where it goes to a minimum, then rises to a local maximum and then eventually goes to zero as $r \to \infty$. The minimum and maximum of $U_{\rm eff}$ should correspond to vanishing derivative, i.e.,

$$-\frac{l^2}{mr^3} + \frac{ar}{b}e^{-r^2/2b} = 0$$

Writing $z = r^2/2b$, this becomes

$$\frac{l^2}{4mab} = z^2\, e^{-z}$$



Left: Plot of U_{eff} for $l^2/(4mab) > 4e^{-2}$ (red) and for $l^2/(4mab) < 4e^{-2}$ (blue). Right: Plot of $z^2 e^{-z}$.

The right hand side, as a function of z, starts at zero, rises to a maximum at $z = 2$ and then falls to zero again. Thus, there is no solution (and hence no bound orbits) if

$$\frac{l^2}{4mab} \geq 4e^{-2}$$

For lower values of $\frac{l^2}{4mab}$, there will be two solutions, z_1, z_2 with $z_2 > z_1$. The value of U_{eff} at these points is

$$U_{\text{eff},1} = \frac{l^2}{4mb}\left(\frac{1}{z_1} - \frac{1}{z_1^2}\right), \qquad U_{\text{eff},2} = \frac{l^2}{4mb}\left(\frac{1}{z_2} - \frac{1}{z_2^2}\right)$$

The range of energy for which we can have bound orbits is thus

$$U_{\text{eff},1} \leq E \leq U_{\text{eff},2}$$

z_1, z_2 have to be obtained by solving a transcendental equation, so we cannot write a more explicit solution.

c) The equation of moltion for r is given by

$$m\ddot{r} = \frac{l^2}{mr^3} - \frac{ar}{b} e^{-r^2/2b}$$

For a circular orbit, we need $\ddot{r} = 0$, which is equivalent to

$$\frac{l^2}{mr^3} - \frac{ar}{b} e^{-r^2/2b} = 0$$

Rewriting this, the relation between the angular momentum and the radius R of the orbit is

$$l^2 = \frac{am}{b} R^4 e^{-R^2/2b}$$

Again, this is a transcendental equation relating l and the radius R.

Problem 3

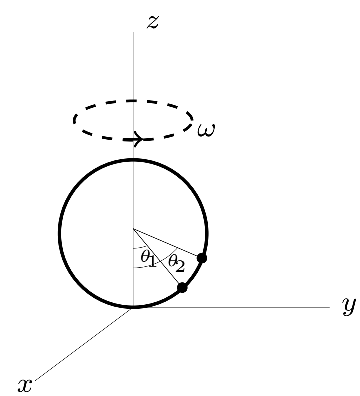
Two beads can move frictionlessly on a vertically placed hoop of radius a. The hoop is rotating around the vertical axis at an angular velocity of ω. Use the angular displacements θ_1 and θ_2 as generalized coordinates. In addition to the gravitational potential, there is also an interaction between the beads given by the potential energy $V_{int} = \frac{1}{2}kd^2$, where d is the separation between the beads and k is a constant. Obtain the Lagrangian and the equations of motion for the system.

__Solution__

The following diagram shows the geometry of the arrangement. The masses of the beads will be designated as m_1, m_2. At a general position for the spinning hoop (at angle $\varphi$with respect to the x-axis), the positions of the beads are given by

$$x_1 = a\sin\theta_1\,\cos\varphi, \qquad y_1 = a\sin\theta_1\,\sin\varphi, \qquad z_1 = a - a\cos\theta_1$$
$$x_2 = a\sin\theta_2\,\cos\varphi, \qquad y_2 = a\sin\theta_2\,\sin\varphi, \qquad z_2 = a - a\cos\theta_2$$

It is easy to check that



$$\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2 = a^2\dot{\theta}_1^2 + a^2\omega^2\sin^2\theta_1$$

where we set $\dot{\varphi} = \omega$. With a similar simplification for the second bead, we can write the kinetic energy as

$$T = \frac{1}{2}m_1a^2(\dot{\theta}_1^2 + \omega^2\sin^2\theta_1) + \frac{1}{2}m_2a^2(\dot{\theta}_2^2 + \omega^2\sin^2\theta_2)$$

The gravitational potential energy is

$$V_{\text{grav}} = -m_1ga\cos\theta_1 - m_2ga\cos\theta_2 + constant$$

The distance between the two beads is given by

$$\begin{aligned} d^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \\ &= a^2\left[(\sin\theta_1 - \sin\theta_2)^2\cos^2\varphi + (\sin\theta_1 - \sin\theta_2)^2\sin^2\varphi + (\cos\theta_1 - \cos\theta_2)^2\right] \\ &= 2a^2\left[1 - \cos(\theta_1 - \theta_2)\right] \end{aligned}$$

The interaction potential energy is thus

$$V_{int} = ka^2\left[1 - \cos(\theta_1 - \theta_2)\right]$$

The Lagrangian for the problem is given by $T - V$ as

$$\begin{aligned}\mathcal{L} =& \frac{a^2}{2}\left[m_1(\dot{\theta}_1^2 + \omega^2\sin^2\theta_1) + m_2(\dot{\theta}_2^2 + \omega^2\sin^2\theta_2)\right]\\ &+ ga(m_1\cos\theta_1 + m_2\cos\theta_2) + ka^2\cos(\theta_1 - \theta_2) + constant\end{aligned}$$

We then find

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = m_1a^2\dot{\theta}_1, \qquad \frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = m_2a^2\dot{\theta}_2\\ &\frac{\partial\mathcal{L}}{\partial\theta_1} = m_1\omega^2a^2\sin\theta_1\cos\theta_1 - m_1ga\sin\theta_1 - ka^2\sin(\theta_1 - \theta_2)\\ &\frac{\partial\mathcal{L}}{\partial\theta_2} = m_2\omega^2a^2\sin\theta_2\cos\theta_2 - m_2ga\sin\theta_2 + ka^2\sin(\theta_1 - \theta_2)\end{aligned}$$

The equations of motion are of the form

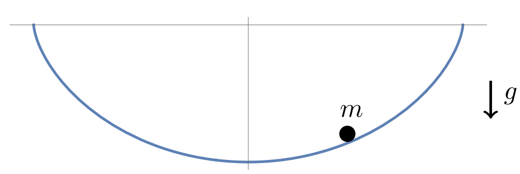
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} = \frac{\partial\mathcal{L}}{\partial q}, \qquad q = \theta_1,\ \theta_2$$

For the present problem, these equations become

$$\begin{aligned}\ddot{\theta}_1 - \omega^2\sin\theta_1\cos\theta_1 &= -\frac{g}{a}\sin\theta_1 - \frac{k}{m_1}\sin(\theta_1 - \theta_2)\\ \ddot{\theta}_2 - \omega^2\sin\theta_2\cos\theta_2 &= -\frac{g}{a}\sin\theta_2 + \frac{k}{m_1}\sin(\theta_1 - \theta_2)\end{aligned}$$

Problem 4

A cycloid is given parametrically by $x = a(\theta + \sin\theta),\ y = -a(1 + \cos\theta)$, where θ is an angular variable, which can be taken to be $-\pi \le \theta \le \pi$. A particle can oscillate around the minimum on the inner surface of a cycloid in the vertical plane, see figure.



Obtain the equations of motion and the general solution. (*Hint*: Use $\sqrt{2a(y + 2a)}$ as the generalized variable. This problem was first analyzed by Christian Huygens and played a role in early attempts to solve the *problem of the longitude*. If you do not know what the problem of the longitude is, do look it up, there is a whole fascinating history for it.)

Solution

Here y is the vertical axis, so that the gravitational potential energy is $V = mgy$. Notice also that we can write $y = -2a + 2a\sin^2(\theta/2)$, so that $\sqrt{2a(y+2a)} = 2a\sin(\theta/2)$. We denote this variable as s. Thus $V = mg(-2a + 2a\sin^2(\theta/2)) = -2amg + (mg/2a)s^2$. Also, from the given formulae,

$$\dot{x} = a\dot{\theta}(1+\cos\theta), \qquad \dot{y} = a\dot{\theta}\,\sin\theta$$

Thus

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) = ma^2\dot{\theta}^2\,(1+\cos\theta) = 2ma^2\,\dot{\theta}^2\,\cos^2(\theta/2)$$

From the definition of s,

$$\dot{s} = \frac{d}{dt}(2a\sin(\theta/2)) = a\dot{\theta}\,\cos(\theta/2)$$

Thus we can write $T = 2m\dot{s}^2$, giving the Lagrangian

$$L = 2m\dot{s}^2 - \frac{mg}{2a}s^2\ + \text{constant}$$

The equation of motion is

$$\ddot{s} + \frac{g}{4a}\,s = 0$$

The solution is simple harmonic motion given by

$$s(t) = s_0\,\sin(\omega t + \varphi_0), \qquad \omega = \sqrt{g/4a}$$

Unlike the case of the simple pendulum, the frequency is independent of the amplitude for all s. (Recall this is true for the simple pendulum only for small amplitudes.) This property is crucial to the argument that suggested the use of the cycloidal oscillations for time measurements and ultimately for identifying the longitude of a place.