PHY 71100: ANALYTICAL DYNAMICS

Problem Set 3

Due October 7, 2024

Problem 1

If there is an ambient cloud of dust (or dark matter) in the solar system, then a planet (of mass m) will be subject to a potential $V = \frac{1}{2}kr^2$, (k is a positive constant), in addition to the Sun's gravitational potential $-GMm/r$.

a) Write down the Lagrangian and the equations of motion for the planet in this case. (You can assume $m \ll M$.)

b) Find the radius for which a circular orbit is possible.

c) Consider a radial perturbation (to first order) of this orbit and work out the perturbation equation. Is the motion bounded in this case?

Solution

a) We have a central potential, so we can drop the θ-dependence from the beginning, using the conservation of the direction of angular momentum. Thus the Lagrangian is given by

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{GMm}{r} - \frac{k}{2}r^2$$

We are taking $m \ll M$, so that the reduced mass $\mu = Mm/(M+m) \approx m$. The equations of motion are given by

$$\begin{aligned} m\ddot{r} &= mr\dot{\varphi}^2 - \frac{GMm}{r^2} - kr \\ \frac{d}{dt}(mr^2\dot{\varphi}) &= 0 \end{aligned}$$

We solve the second equation in the usual way by setting $mr^2\dot{\varphi} = l$, the magnitude of angular momentum. The radial equation then becomes

$$m\ddot{r} = \frac{l^2}{mr^3} - \frac{GMm}{r^2} - kr$$

b) For a circular orbit we need $\dot{r} = 0$ and $\ddot{r} = 0$. Thus the right hand side of the equation given above must vanish. This gives the radius R of the circular orbit as the solution of

$$\frac{l^2}{m} - GMmR - kR^4 = 0 \tag{1}$$

This seems difficult to solve since it involves the fourth power of R. We will come back to this question.

c) Now consider a radial perturbation $r = R + \eta$. Using

$$\begin{aligned}
\frac{l^2}{m(R+\eta)^3} &= \frac{l^2}{mR^3} - \frac{3l^2}{mR^4}\,\eta + \mathcal{O}(\eta^2) \\
\frac{GMm}{(R+\eta)^2} &= \frac{GMm}{R^2} - \frac{2GMm}{R^3}\,\eta + \mathcal{O}(\eta^2) \\
k(R+\eta) &= kR + k\eta
\end{aligned}$$

we find

$$m\ddot{\eta} = -\left(\frac{3l^2}{mR^4} - \frac{2GMm}{R^3} + k\right)\eta$$

where we used (1). We eliminate the l^2-dependent term using (1) again to get

$$m\ddot{\eta} = -\left(\frac{GMm}{R^3} + 4k\right)\eta$$

Now define

$$\Omega = \sqrt{\frac{GMm}{R^3} + 4k}$$

This is real since the quantity inside the square root sign is positive. The solution to this equation is then of the form

$$\eta = A\,\sin[\Omega(t - t_0)]$$

The perturbation is clearly bounded since $|\sin| \leq 1$.

Returning to the problem of solving (1), the difficulty is in solving in terms of l. Since r is fixed, $l = mR^2\omega$, where $\dot{\varphi} = \omega$. Notice that ω is related to the period of revolution T by $\omega = 2\pi/T$. In terms of ω, (1) becomes

$$(m\omega^2 - k)R = \frac{GMm}{R^2}$$

with the solution

$$R = \left(\frac{GMm}{m\omega^2 - k}\right)^{\frac{1}{3}}$$

A word of warning: The use of ω^2 before you do the perturbation is not right, since ω is not preserved in the perturbation. The angular momentum l is, so the change of r is compensated by change of ω. So, for the perturbed equation, we must use fixed l and change r to $R + \eta$.

Problem 2

In class (and in my lecture notes) I worked out the scattering for a particle subject to a repulsive $1/r^2$ potential. Here we will consider an attractive central force potential

$$V(r) = -\frac{\alpha}{r^2}, \qquad \alpha > 0$$

a) Show that the particle will fall into the center for angular momenta below a critical value. Find this value.
b) For energies and angular momenta for which there is no such instability, find the differential scattering cross section for particles scattering off this potential.

Solution

a) Once again, this is a central force problem, so we can drop the θ-dependence. The Lagrangian is then

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{\alpha}{r^2}$$

The equations of motion are

$$\begin{aligned} m\ddot{r} &= mr\dot{\varphi}^2 - \frac{2\alpha}{r^3} \\ \frac{d}{dt}(mr^2\dot{\varphi}) &= 0 \end{aligned}$$

As before, we set $l = mr^2\dot{\varphi}$ as the solution of the second equation and reduce the first one to

$$m\ddot{r} = \left(\frac{l^2 - 2m\alpha}{m}\right)\frac{1}{r^3}$$

If $l^2 < 2m\alpha$, the right hand side is negative, the radial acceleration is negative and the particle falls into the center, the rate of fall increasing as the radius decreases. Thus we have an instability (of capture of the incident particle by the scattering center) for $l^2 < 2m\alpha$.

b) Writing $u = 1/r$, the orbit equation is

$$l\,\frac{du}{d\varphi} = \mp\sqrt{2mE - (l^2 - 2m\alpha)u^2}$$

The sign only tells us whether the particle is traveling in the clockwise or anti-clockwise direction; it is irrelevant for the scattering cross section. So we will take take the positive sign on the right hand side of this equation. To avoid the instability, we take $l^2 > 2m\alpha$ and $E > 0$. Thus

$$\int \frac{l du}{\sqrt{2mE - (l^2 - 2m\alpha)u^2}} = \varphi$$

We use the substitution $u = \sqrt{2mE}\sin f$ to carry out the integration and get

$$\frac{1}{r} = \sqrt{\frac{2mE}{l^2 - 2m\alpha}}\,\sin\left(\sqrt{\frac{l^2 - 2m\alpha}{l^2}}\varphi\right)$$

The solutions for $r \to \infty$ are

$$\varphi = 0, \quad \pi\sqrt{\frac{l^2}{l^2 - 2m\alpha}}$$

Notice that the second value is greater than π. So we can think of the process as the particle coming in along the positive x-axis ($\varphi = 0$) towards the origin and scattered to an angle $\pi\sqrt{\frac{l^2}{l^2-2m\alpha}}$. The continuation of the incoming line is along the negative x-axis, i.e., at angle π. The scattering angle is thus

$$\theta = \pi\sqrt{\frac{l^2}{l^2-2m\alpha}} - \pi$$

Writing $l = mv\rho$, we solve for the impact parameter ρ as

$$\rho = \sqrt{\frac{\alpha}{E}}\,\frac{\pi+\theta}{\sqrt{\theta(2\pi+\theta)}}$$

The cross section is then given as

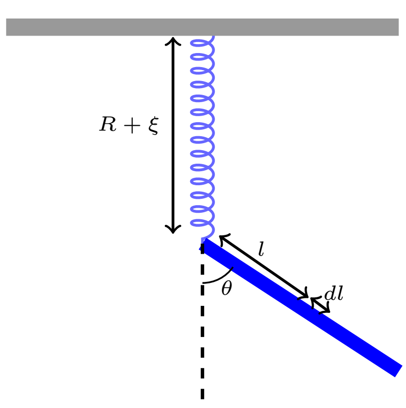
$$\begin{aligned} d\sigma &\equiv \rho\left|\frac{d\rho}{d\theta}\right|\frac{1}{\sin\theta}\,d\Omega \\ &= \left(\frac{\alpha}{E}\right)\frac{\pi^2(\pi+\theta)}{\theta^2(2\pi+\theta)^2}\,\frac{1}{\sin\theta}\,d\Omega \end{aligned}$$

If you compare this with the result for the repulsive case given in the lecture notes, you will see that the difference is in the sign of θ.

Problem 3

A bar of length L and uniform mass per unit length is suspended from the lower end of a vertically suspended spring, of negligible mass and spring constant k, as shown. The spring can only oscillate in the vertical direction and the motion is confined to one vertical plane, see figure. Obtain the Lagrangian and the equations of motion for the system without making any approximation of small amplitudes. (Hint: Consider a differential mass element of mass $(M/L)dl$, where M is the total mass of the bar. Obtain the kinetic and potential energies for this mass element and integrate over l from zero to L to get the energies for the bar.)

Solution



Problem 3

The set-up of the problem is as shown in figure. $R+\xi$ denotes the length of the spring, R being the equilibrium length and ξ being the extension (positive or negative). It is given that the spring remains vertical, although it can oscillate in the vertical extension. The bar is taken to have length L. Consider a small element of the bar with mass $(M/L)dl$ at a distance l from the end of the spring. The position of this element is given by

$$x = l\,\sin\theta, \qquad y = -\left[R+\xi+l\,\cos\theta\right]$$

We find

$$\dot{x} = l\,\dot{\theta}\,\cos\theta, \qquad \dot{y} = -\dot{\xi}+l\,\dot{\theta}\,\sin\theta$$

The kinetic energy for this mass element is then given by

$$\begin{aligned} T &= \frac{1}{2}(M dl/L)\left[\dot{x}^2+\dot{y}^2\right] \\ &= (M/2L)dl\,\left[\dot{\theta}^2 l^2(\cos^2\theta+\sin^2\theta)+\dot{\xi}^2-2\dot{\xi}\dot{\theta}\,l\,\sin\theta\right] \\ &= (M/2L)dl\,\left[l^2\dot{\theta}^2+\dot{\xi}^2-2\dot{\xi}\dot{\theta}\,l\sin\theta\right] \end{aligned}$$

To get the total kinetic energy, we must integrate this over all mass elements, i.e., from $l=0$ to L. This gives

$$T = \frac{ML^2}{6}\dot{\theta}^2+\frac{M}{2}\dot{\xi}^2-\frac{ML}{2}\dot{\xi}\dot{\theta}\,\sin\theta$$

The gravitational potential energy of the mass element is $(M/L)dlgy$ and there is also $\frac{1}{2}k\xi^2$ from stretching the spring. Integrating the first one over l, we find

$$V = -Mg\left[(R+\xi)+\frac{L}{2}\cos\theta\right]+\frac{1}{2}k\xi^2$$

The Lagrangian is

$$L = T-V = \frac{ML^2}{6}\dot{\theta}^2+\frac{M}{2}\dot{\xi}^2-\frac{ML}{2}\dot{\xi}\dot{\theta}\,\sin\theta+Mg\left[(R+\xi)+\frac{L}{2}\cos\theta\right]-\frac{1}{2}k\xi^2$$

This gives:

$$\begin{aligned} \frac{\partial L}{\partial\dot{\theta}} &= \frac{ML^2}{3}\dot{\theta}-\frac{ML}{2}\dot{\xi}\sin\theta, \qquad & \frac{\partial L}{\partial\dot{\xi}} &= M\dot{\xi}-\frac{ML}{2}\dot{\theta}\sin\theta \\ \frac{\partial L}{\partial\theta} &= -\frac{ML}{2}\dot{\xi}\dot{\theta}\cos\theta-\frac{MgL}{2}\sin\theta, \qquad & \frac{\partial L}{\partial\xi} &= Mg-k\xi \end{aligned}$$

The equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\xi}}\right) = \frac{\partial L}{\partial\xi},$$

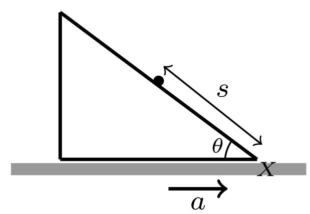
From the derivatives we calculated, these become

$$
\begin{aligned}
\frac{ML^2}{3}\ddot{\theta} - \frac{ML}{2}\ddot{\xi}\,\sin\theta &= -\frac{MgL}{2}\sin\theta \\
M\ddot{\xi} - \frac{ML}{2}\ddot{\theta}\sin\theta - \frac{ML}{2}\dot{\theta}^2\cos\theta &= Mg - k\xi
\end{aligned}
$$

These are the equations of motion for the problem.

Problem 4

A particle of mass m can slide down an inclined plane under the force of gravity. This entire setup is placed inside a train (the bed of the train is the thickened line in figure) which is moving with a constant acceleration in a straight line, say, along the x-axis. Obtain the Lagrangian and the equations of motion for the particle.



Problem 4

Solution

We take the distance along the inclined plane as s, taken from the end point marked as X in the diagram. Since the plane is accelerating, X moves with acceleration. The coordinates of the mass are given by

$$x = X - s\,\cos\theta, \qquad y = s\,\sin\theta$$

Thus

$$\dot{x} = \dot{X} - \dot{s}\cos\theta, \qquad \dot{y} = \dot{s}\sin\theta$$

The angle θ is the fixed angle of inclination, so it does not change with time. The kinetic energy of the mass is given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left(\dot{s}^2 - 2\dot{s}\dot{X}\,\cos\theta + \dot{X}^2\right)$$

The potential energy is given as $V = mgy$. Thus

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left(\dot{s}^2 - 2\dot{s}\dot{X}\,\cos\theta\right) - mgs\sin\theta\, + \frac{1}{2}m\dot{X}^2$$

The last term does not contribute to the equations of motion since X is not dynamically determined. It is given that it is externally controlled to be moving with acceleration a. The only dynamical variable is s. We get

$$\frac{\partial \mathcal{L}}{\partial \dot{s}} = m\dot{s} - m\dot{X}\cos\theta, \qquad \frac{\partial \mathcal{L}}{\partial s} = mg\sin\theta$$

The equation of motion is thus

$$\frac{d}{dt}\left(m\dot{s} - m\dot{X}\cos\theta\right) = mg\sin\theta$$

Since $\ddot{X} = a$, this simplifies to

$$\ddot{s} = g\sin\theta + a\cos\theta$$