PHY 71100: ANALYTICAL DYNAMICS

Problem Set 4

Due November 6, 2024

Problem 1

As discussed in class, a general $SU(2)$ matrix can be written as $U = \exp(\frac{i}{2}\sigma_k\theta_k)$ where σ_k are the Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

You can verify easily that these obey the algebra

$$\sigma_i\sigma_j = \delta_{ij}\,\mathbb{1} + i\sum_k \epsilon_{ijk}\,\sigma_k$$

(Here $\mathbb{1}$ denotes the 2×2 identity matrix.) The relation between an $SU(2)$ matrix and the corresponding $SO(3)$ rotation matrix was obtained in class as

$$U^{-1}\sigma_i\,U = R_{ij}(\theta)\,\sigma_j \tag{1}$$

a) Choose the specific matrix $U = \exp(\frac{i}{2}\sigma_1\theta_1)$. Expand the exponential to write this as a 2×2 matrix.

b) Calculate the corresponding rotation matrix $R^{(1)}_{ij}(\theta_1)$ using the formula (1).

Solution

a) U is a function (specifically the exponential) of a matrix, so it is defined by a power series expansion. With the given σ_1,

$$\sigma_1^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}, \qquad \sigma_1^3 = \sigma_1, \ \text{etc.}$$

Thus

$$\begin{aligned} U &= \mathbb{1} + i(\theta_1/2)\sigma_1 + i^2\frac{(\theta_1/2)^2}{2!}\mathbb{1} + i^3\frac{(\theta_1/2)^3}{3!}\sigma_1 + \cdots \\ &= \begin{pmatrix} 1 - \frac{1}{2!}(\theta_1/2)^2 + \frac{1}{4!}(\theta_1/2)^4 + \cdots & i\left[(\theta_1/2) - \frac{1}{3!}(\theta_1/2)^3 + \cdots\right] \\ i\left[(\theta_1/2) - \frac{1}{3!}(\theta_1/2)^3 + \cdots\right] & 1 - \frac{1}{2!}(\theta_1/2)^2 + \frac{1}{4!}(\theta_1/2)^4 + \cdots \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta_1/2) & i\sin(\theta_1/2) \\ i\sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix} \end{aligned}$$

b) The inverse matrix is given by $U^\dagger$ since it is unitary; it is also the same as $U(-\theta_1)$. Thus

$$\begin{aligned}
U^{-1}\sigma_1 U &= \begin{pmatrix} \cos(\theta_1/2) & -i\sin(\theta_1/2) \\ -i\sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta_1/2) & i\sin(\theta_1/2) \\ i\sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix} \\
&= \begin{pmatrix} -i\sin(\theta_1/2) & \cos(\theta_1/2) \\ \cos(\theta_1/2) & -i\sin(\theta_1/2) \end{pmatrix} \begin{pmatrix} \cos(\theta_1/2) & i\sin(\theta_1/2) \\ i\sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \sigma_1 \\
U^{-1}\sigma_2 U &= \begin{pmatrix} \cos(\theta_1/2) & -i\sin(\theta_1/2) \\ -i\sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta_1/2) & i\sin(\theta_1/2) \\ i\sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix} \\
&= \begin{pmatrix} 2\sin(\theta_1/2)\cos(\theta_1/2) & -i(\cos^2(\theta_1/2) - \sin^2(\theta_1/2)) \\ i(\cos^2(\theta_1/2) - \sin^2(\theta_1/2)) & -2\sin(\theta_1/2)\cos(\theta_1/2) \end{pmatrix} \\
&= \begin{pmatrix} \sin\theta_1 & -i\cos\theta_1 \\ i\cos\theta_1 & -\sin\theta_1 \end{pmatrix} = \sin\theta_1\,\sigma_3 + \cos\theta_1\,\sigma_2 \\
U^{-1}\sigma_3 U &= \begin{pmatrix} \cos(\theta_1/2) & -i\sin(\theta_1/2) \\ -i\sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\theta_1/2) & i\sin(\theta_1/2) \\ i\sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_1/2) & i\sin(\theta_1/2) \\ -i\sin(\theta_1/2) & -\cos(\theta_1/2) \end{pmatrix} \begin{pmatrix} \cos(\theta_1/2) & i\sin(\theta_1/2) \\ i\sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta_1 & i\sin\theta_1 \\ -i\sin\theta_1 & -\cos\theta_1 \end{pmatrix} = -\sin\theta_1\,\sigma_2 + \cos\theta_1\,\sigma_3
\end{aligned}$$

From the given equation (1), these should be

$$\begin{aligned}
U^{-1}\sigma_1 U &= R_{11}\sigma_1 + R_{12}\sigma_2 + R_{13}\sigma_3 \\
U^{-1}\sigma_2 U &= R_{21}\sigma_1 + R_{22}\sigma_2 + R_{23}\sigma_3 \\
U^{-1}\sigma_3 U &= R_{31}\sigma_1 + R_{32}\sigma_2 + R_{33}\sigma_3
\end{aligned}$$

Comparing, we identify

$$R_{11} = 1, \quad R_{12} = R_{13} = 0, \quad R_{21} = 0, \quad R_{22} = \cos\theta_1, \quad R_{23} = \sin\theta_1,$$
$$R_{31} = 0, \quad R_{32} = -\sin\theta_1, \quad R_{33} = \cos\theta_1$$

This can be written out as a 3×3 matrix:

$$R^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

Problem 2

In class, and in my lecture notes, I gave the formula for the rotation matrices $R^{(1)}_{ij}$, $R^{(2)}_{ij}$, $R^{(3)}_{ij}$ around the three Cartesian coordinate directions x_1, x_2 and x_3. Consider small angles ($\theta \approx \alpha \ll 1$) and write them as

$$R \approx \mathbb{1} + \alpha \cdot J$$

(Here $\mathbb{1}$ denotes the 3×3 identity matrix.)

a) Identify the three matrices J corresponding to the three rotation matrices.

b) Work out, by explicit matrix multiplication, the commutators $J^{(1)}J^{(2)} - J^{(2)}J^{(1)}$, $J^{(2)}J^{(3)} - J^{(3)}J^{(2)}$, $J^{(3)}J^{(1)} - J^{(1)}J^{(3)}$. (Do you recognize these from quantum mechanics?)

Solution

a) The three rotation matrices are

$$R^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix}, \quad R^{(2)} = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}, \quad R^{(3)} = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Taking θ_i to be small and expanding $\cos\theta_i \approx 1$, $\sin\theta_i \approx \theta_i$, we find $R^{(k)} \approx \mathbb{1} + \theta_k J^{(k)}$ with

$$J^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad J^{(2)} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad J^{(3)} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b) By direct calculation

$$\begin{aligned} J^{(1)}J^{(2)} - J^{(2)}J^{(1)} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -J^{(3)} \\ J^{(2)}J^{(3)} - J^{(3)}J^{(2)} &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = -J^{(1)} \end{aligned}$$

$$J^{(3)}J^{(1)} - J^{(1)}J^{(3)} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = -J^{(2)}$$

These may be written as

$$[J^{(i)}, J^{(j)}] = -\epsilon^{ijk}\, J^{(k)}$$

These are essentially the commutation rules for angular momentum in quantum mechanics. To make the connection more explicit, define $L^i = -i\hbar J^{(i)}$. Then we get

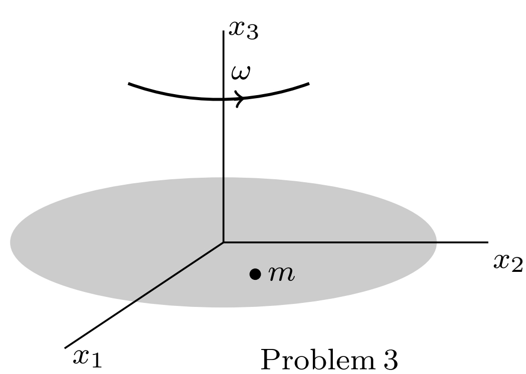
$$[L^i, L^j] = i\hbar\, \epsilon^{ijk}\, L^k$$

Problem 3

A disk is in the horizontal plane and is rotating around the vertical axis with angular velocity ω, see figure. (Think of this as a merry-go-round.) A bead of mass m is sliding on the disk, moving from the center to the periphery. Write down the Lagrangian and the equations of motion for the bead in the rotating frame using cylindrical coordinates

$$x_1 = r\,\cos\varphi, \qquad x_2 = r\,\sin\varphi, \qquad x_3 = x_3$$

You must simplify the Coriolis and centrifugal terms with the appropriate choice of coordinates and angular velocity. *You do not have to solve these equations, although they are rather straightforward to solve.*



Problem 3

Solution

The Lagrangian for particle motion in a rotating frame was derived in class as

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 - 2\epsilon_{ijk}\dot{x}^i x^j \omega^k + \omega^2 x^2 - (\omega \cdot x)^2\right) - V(x)$$

For the present case, x_3 does not change, so the potential energy $V = mgx_3$ can be taken to be a constant. Further, $\vec{\omega} = (0, 0, \omega)$ since there is only rotation around the third axis. We find

$$\begin{aligned}
\dot{x}_1^2 + \dot{x}_2^2 &= (\dot{r}\cos\varphi - \dot{\varphi}\, r\sin\varphi)^2 + (\dot{r}\sin\varphi + \dot{\varphi}\, r\cos\varphi)^2 = \dot{r}^2 + r^2\dot{\varphi}^2 \\
-2\epsilon_{ijk}\dot{x}^i x^j \omega^k &= -2(\dot{x}^1 x^2 - \dot{x}^2 x^1)\omega \\
&= -2\omega\left[(\dot{r}\cos\varphi - \dot{\varphi}\, r\sin\varphi) r\sin\varphi - (\dot{r}\sin\varphi + \dot{\varphi}\, r\cos\varphi) r\cos\varphi\right] = 2\,\omega r^2\dot{\varphi} \\
\omega^2 x^2 - (\omega\cdot x)^2 &= \omega^2(x_1^2 + x_2^2 + x_3^2) - (\omega x_3)^2 = \omega^2(x_1^2 + x_2^2) = \omega^2 r^2
\end{aligned}$$

Th Lagrangian is thus given by

$$\mathcal{L} = \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\varphi}^2 + 2\,\omega r^2\dot{\varphi} + \omega^2 r^2\right] + \text{constant}$$

We then get

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{r}} &= m\dot{r}, \qquad \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = mr^2(\dot{\varphi} + \omega), \qquad \frac{\partial\mathcal{L}}{\partial\varphi} = 0 \\
\frac{\partial\mathcal{L}}{\partial r} &= mr\dot{\varphi}^2 + 2mr\omega\dot{\varphi} + m\omega^2 r = mr(\dot{\varphi} + \omega)^2
\end{aligned}$$

The equations of motion are:

$$\ddot{r} = r(\dot{\varphi} + \omega)^2, \qquad \frac{d}{dt}\left[r^2(\dot{\varphi} + \omega)\right] = 0$$