PHY 71100: ANALYTICAL DYNAMICS

Problem Set 4

Due November 6, 2024

Problem 1

As discussed in class, a general $SU(2)$ matrix can be written as $U = \exp(\frac{i}{2}\sigma_k\theta_k)$ where σ_k are the Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

You can verify easily that these obey the algebra

$$\sigma_i\sigma_j = \delta_{ij}\,\mathbb{1} + i\,\sum_k \epsilon_{ijk}\,\sigma_k$$

(Here $\mathbb{1}$ denotes the 2×2 identity matrix.) The relation between an $SU(2)$ matrix and the corresponding $SO(3)$ rotation matrix was obtained in class as

$$U^{-1}\sigma_i\,U = R_{ij}(\theta)\,\sigma_j \tag{1}$$

a) Choose the specific matrix $U = \exp(\frac{i}{2}\sigma_1\theta_1)$. Expand the exponential to write this as a 2×2 matrix.

b) Calculate the corresponding rotation matrix $R^{(1)}_{ij}(\theta_1)$ using the formula (1).

Problem 2

In class, and in my lecture notes, I gave the formula for the rotation matrices $R^{(1)}_{ij}$, $R^{(2)}_{ij}$, $R^{(3)}_{ij}$ around the three Cartesian coordinate directions x_1, x_2 and x_3. Consider small angles ($\theta \approx \alpha \ll 1$) and write them as

$$R \approx \mathbb{1} + \alpha\cdot J$$

(Here $\mathbb{1}$ denotes the 3×3 identity matrix.)

a) Identify the three matrices J corresponding to the three rotation matrices.

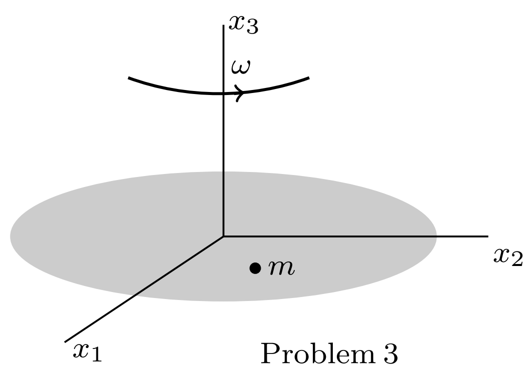
b) Work out, by explicit matrix multiplication, the commutators $J^{(1)}J^{(2)} - J^{(2)}J^{(1)}$, $J^{(2)}J^{(3)} - J^{(3)}J^{(2)}$, $J^{(3)}J^{(1)} - J^{(1)}J^{(3)}$. (Do you recognize these from quantum mechanics?)

Problem 3

A disk is in the horizontal plane and is rotating around the vertical axis with angular velocity ω, see figure. (Think of this as a merry-go-round.) A bead of mass m is sliding on the disk, moving from the center to the periphery. Write down the Lagrangian and the equations of motion for the bead in the rotating frame using cylindrical coordinates

$$x_1 = r\ \cos\varphi, \quad x_2 = r\ \sin\varphi, \quad x_3 = x_3$$

You must simplify the Coriolis and centrifugal terms with the appropriate choice of coordinates and angular velocity. *You do not have to solve these equations, although they are rather straightforward to solve.*



Problem 3