PHY 71100: ANALYTICAL DYNAMICS

Problem Set 5

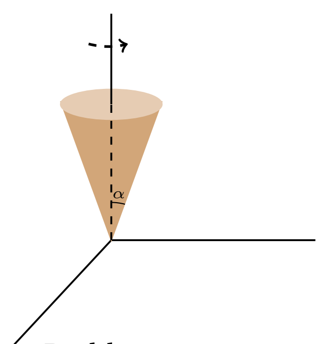
Due November 20, 2024

Problem 1

Consider a solid cone which is rotating around its axis, as shown. Calculate the moment of inertia and obtain the Lagrangian for this motion.

Solution

Let α denote the opening angle of the cone, as shown in figure. We use λ to denote the radius of the cone at a height z. Then we can write $z = \lambda \tan\alpha$. Let ρ denote the mass



Problem 1

density of the material of the cone and R denote the radius of the base of the cone. For rotations around the z-axis, we need I_{33}. This is given by

$$\begin{aligned} I_{33} &= \int d^3x\, \rho\left[(x^2+y^2+z^2)\delta_{33} - zz\right] = \int dzrdrd\varphi\, \rho\, r^2 \\ &= 2\pi\rho\tan\alpha \int_0^R d\lambda \int_0^\lambda drr^3 = 2\pi\rho\tan\alpha \int_0^R d\lambda \frac{\lambda^4}{4} = \pi\rho\tan\alpha\, \frac{R^5}{10} \end{aligned}$$

The mass of the cone is given by

$$M = \int dzdrrd\varphi\, \rho = 2\pi\rho\tan\alpha \int_0^R d\lambda \int_0^\lambda drr = 2\pi\rho\tan\alpha \int_0^R d\lambda \frac{\lambda^2}{2} = \pi\rho\tan\alpha\, \frac{R^3}{3}$$

Using this

$$I_{33} = \frac{3}{10} M R^2$$

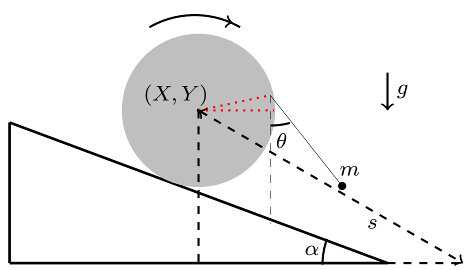
Problem 2

A solid circular disk of mass M and radius R can roll down an inclined plane under the force of gravity. There is an axle at the rim of the disk from which is suspended a pendulum, with a bob of mass m and a string of negligible mass. (The axle is to ensure that the string for the pendulum does not wind around the disk, but keeps a fixed length, as it rolls down.) Obtain the Lagrangian for the motion of the pendulum and the disk.

Solution

The figure illustrates the geometry of the set-up. Let (X, Y) denote the position of the center of the disk and let s be the distance from this point to the right lower edge of the inclined plane measured along the plane. Then we have

$$(X, Y) = (-s\cos\alpha, s\sin\alpha)$$



Problem 2

Relative to this center of the disk, the point of suspension is displaced by $(R\cos\varphi, R\sin\varphi)$, where φ is the angle between the two dotted red lines in the figure. So the the coordinates of the point of suspension are $(X + R\cos\varphi, Y + R\sin\varphi)$. The position of the bob of the pendulum is then given by

$$(x, y) = (-s\cos\alpha + R\cos\varphi + l\sin\theta, s\sin\alpha + R\sin\varphi - l\cos\theta)$$

Since we have rolling without slipping, we also have $\dot{s} = R\dot{\varphi}$. We thus get

$$\begin{aligned} \dot{X} &= -\dot{s}\cos\alpha = -R\dot{\varphi}\cos\alpha, \qquad \dot{Y} = \dot{s}\sin\alpha = R\dot{\varphi}\sin\alpha \\ \dot{x} &= -\dot{s}\cos\alpha - \dot{\varphi}\,R\sin\varphi + \dot{\theta}\,l\cos\theta = -R\dot{\varphi}(\cos\alpha + \sin\varphi) + l\dot{\theta}\cos\theta \\ \dot{y} &= \dot{s}\sin\alpha + \dot{\varphi}\,R\cos\varphi + \dot{\theta}\,l\sin\theta = R\dot{\varphi}(\sin\alpha + \cos\varphi) + l\dot{\theta}\sin\theta \end{aligned}$$

Thus

$$\dot{X}^2 + \dot{Y}^2 = R^2\dot{\varphi}^2$$

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 &= R^2\dot{\varphi}^2\left[(\cos\alpha + \sin\varphi)^2 + (\sin\alpha + \cos\varphi)^2\right] + l^2\dot{\theta}^2 \\ &\quad + 2Rl\dot{\theta}\dot{\varphi}\left[\sin\theta(\sin\alpha + \cos\varphi) - \cos\theta(\cos\alpha + \sin\varphi)\right] \\ &= 2R^2\dot{\varphi}^2(1 + \sin(\varphi + \alpha)) + l^2\dot{\theta}^2 - 2Rl\dot{\theta}\dot{\varphi}\cos(\theta + \alpha) + 2Rl\dot{\theta}\dot{\varphi}\sin(\theta - \varphi) \end{aligned}$$

The moment of inertia for the disk is $\frac{1}{2}MR^2$. Thus the kinetic energy is obtained as

$$T_{\rm disk} = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}I\dot{\varphi}^2 = \frac{3}{4}MR^2\dot{\varphi}^2$$

$$\begin{aligned} T_{\rm pend} &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= mR^2\dot{\varphi}^2\ (1 + \sin(\alpha+\varphi)) + mRl\,\dot{\theta}\dot{\varphi}\ [\sin(\theta-\varphi) - \cos(\theta+\alpha)] \\ &\quad + \frac{1}{2}ml^2\dot{\theta}^2 \end{aligned}$$

The potential energy is given by $MgY + mgy$, i.e.,

$$V = (M+m)g\sin\alpha\,(R\varphi) + mgR\sin\varphi - mgl\cos\theta$$

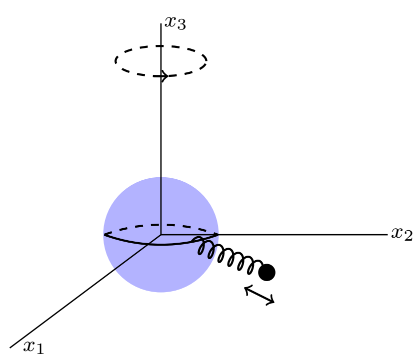
The Lagrangian is given by $L = T_{\rm disk} + T_{\rm pend} - V$ with $T_{\rm disk}$, $T_{\rm pend}$ and V as above.

Problem 3

A solid spherical ball of mass M and radius R is at the center of the coordinate system as shown. At one point on the equator is attached a spring (of negligible mass) which is horizontal, in the (x, y)-plane. At the other end of the spring is a small mass m. The system can undergo rotations around the z-axis and the mass m can move by stretching (or compressing) the spring. (Ignore any other kind of motion for the spring.)

a) Obtain the Lagrangian for the system and the equations of motion. Show that there is a conserved angular momentum for the system.

b) If l_0 is the length of the unstretched spring when it is at rest, determine its equilibrium length when the system is rotating with angular momentum J, in the approximation $M \gg m$. (Moment of inertia for a sphere $= \frac{2}{5}MR^2$.)



Problem 3

Solution

We first find the coordinates of the mass m at the end of the spring. Taking the length of the spring as l, we find

$$x = (R + l)\cos\varphi, \qquad y = (R + l)\sin\varphi$$

This gives

$$\dot{x} = -(R + l)\sin\varphi\,\dot{\varphi} + \dot{l}\cos\varphi, \quad \dot{y} = (R + l)\cos\varphi\,\dot{\varphi} + \dot{l}\sin\varphi$$

In principle, we can have a type of motion where the spring can sway at the edge of the ball, so that the angular variable for the mass m and the rotation angle for the ball are different. But here we are told that only the radial motion of extension of spring need be considered. So we have a common angular velocity $\dot{\varphi}$. Thus

$$\begin{aligned} T &= \frac{1}{2}I\dot{\varphi}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}I\dot{\varphi}^2 + \frac{m}{2}(R + l)^2\dot{\varphi}^2 + \frac{m}{2}\dot{l}^2 \\ &= \frac{1}{2}\left(I + m(R + l)^2\right)\dot{\varphi}^2 + \frac{1}{2}m\,\dot{l}^2 \end{aligned}$$

The potential energy is $V = \frac{1}{2}k(l - l_0)^2$, giving

$$L = \frac{1}{2}\left(I + m(R + l)^2\right)\dot{\varphi}^2 + \frac{1}{2}m\,\dot{l}^2 - \frac{k}{2}(l - l_0)^2$$

Notice that there is no explicit φ-dependence, so we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\varphi}}\right) = 0 \quad \Longrightarrow \quad J = \left(\frac{\partial L}{\partial\dot{\varphi}}\right) = \left[I + m(R + l)^2\right]\dot{\varphi} = \text{constant}$$

This quantity J is the conserved angular momentum for this case. The equation of motion for l is given by

$$m\ddot{l} = m(R + l)\dot{\varphi}^2 - k(l - l_0)$$

Using J, this becomes

$$\ddot{l} = \frac{(R + l)J^2}{[I + m(R + l)^2]^2} - \frac{k}{m}(l - l_0)$$

The equilibrium point corresponding to $\ddot{l} = 0$ is given by $l = l_*$, with

$$\frac{(R + l_*)J^2}{[I + m(R + l_*)^2]^2} - \frac{k}{m}(l_* - l_0) = 0$$

The moment of inertia for the sphere is $I = (2/5)MR^2$, so we can solve the equation for l_* in the approximation of large M, or $I \gg m(R + l_*)^2$, as

$$l_* = l_0 + (R + l_0)\left[\frac{J^2 m}{kI^2 - J^2 m}\right]$$

Problem 4

Consider a thin homogeneous plate with the principal moments of inertia I_1, I_2, I_3,

$I_2 > I_1 = I_2 \cos 2\alpha$ and $I_3 = I_1 + I_2$. Take the origin of the body-fixed and space-fixed coordinates as the center of mass of the plate, with (x_1, x_2)-axes along the plate and the x_3-axis perpendicular to it. At time $t = 0$, the plate starts rotating freely with angular velocity Ω about an axis at an angle α from the plane of the plate and perpendicular to the x_2-axis, so that initially $\Omega_1(0) = \Omega \cos\alpha,\ \Omega_2(0) = 0,\ \Omega_3(0) = \Omega \sin\alpha$. Show that the angular velocity about the x_2-axis is given by

$$\Omega_2(t) = \Omega \cos\alpha\ \tanh(\Omega t \sin\alpha)$$

<u>**Solution**</u>

The given relations among the moments of inertia lead to

$$I_3 = I_1 + I_2 = I_2(1+\cos 2\alpha) = 2I_2\cos^2\alpha, \qquad I_2 - I_1 = I_2(1-\cos 2\alpha) = 2I_2 \sin^2\alpha$$

The Euler equations for the free (torque-free) rotation of the rigid body are

$$\begin{aligned} I_1\dot{\Omega}_1 + (I_3 - I_2)\Omega_2\Omega_3 &= 0 \\ I_2\dot{\Omega}_2 + (I_1 - I_3)\Omega_3\Omega_1 &= 0 \\ I_3\dot{\Omega}_3 + (I_2 - I_1)\Omega_1\Omega_2 &= 0 \end{aligned}$$

Using the given relations, these simplify as

$$\dot{\Omega}_1 + \Omega_2\Omega_3 = 0 \tag{1}$$

$$\dot{\Omega}_2 - \Omega_3\Omega_1 = 0 \tag{2}$$

$$\dot{\Omega}_3 + (\tan^2\alpha)\,\Omega_1\Omega_2 = 0 \tag{3}$$

From (1), (2), we get

$$\frac{d}{dt}(\Omega_1^2 + \Omega_2^2) = 0 \implies \Omega_1^2 + \Omega_2^2 = \text{constant, say } C$$

From (1), (3), we get

$$\frac{d}{dt}(\Omega_3^2 - \tan^2\alpha\,\Omega_1^2) = 0 \implies \Omega_3^2 - \tan^2\alpha\,\Omega_1^2 = \text{constant}' = C'$$

Initially, we have $\Omega_1(0) = \Omega\cos\alpha$, $\Omega_3(0) = \Omega\sin\alpha$, so that $C' = 0$. Thus for all time, $\Omega_3 = \Omega_1 \tan\alpha$. Also, initially we have $\Omega_2(0) = 0$ which gives $C = \Omega_1(0)^2 = \Omega^2\cos^2\alpha$. Now we can use (2) to solve for Ω_2,

$$\dot{\Omega}_2 = \Omega_3\Omega_1 = \Omega_1\tan\alpha\,\Omega_1 = \tan\alpha\left(C - \Omega_2^2\right) = \tan\alpha\left(\Omega^2\cos^2\alpha - \Omega_2^2\right)$$

We can integrate as

$$\int \frac{d\Omega_2}{\Omega^2 \cos^2 \alpha - \Omega_2^2} = \tan \alpha \int dt$$

This can be done using the substitution $\Omega_2 = (\Omega \cos \alpha)\, u$ and

$$\int \frac{du}{1 - u^2} = \tanh^{-1} u$$

So we get

$$\tanh^{-1} u = \Omega t \sin \alpha \implies \Omega_2 = \Omega \cos \alpha \, \tanh\big[\Omega t \sin \alpha\big]$$