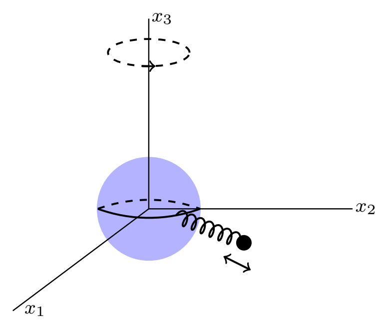
PHY 71100: ANALYTICAL DYNAMICS

Problem Set 6

Due December 2, 2024

Problem 1 (10 points)

A solid spherical ball of mass M and radius R is at the center of the coordinate system as shown. At one point on the equator is attached a spring (of negligible mass) which is horizontal, in the (x, y)-plane. At the other end of the spring is a small mass m. The system can undergo rotations around the z-axis and the mass m can move by stretching (or compressing) the spring. (Ignore any other kind of motion for the spring.) The Lagrangian for this was obtained in one of the previous problem sets as



Problem 1

$$L = \frac{1}{2}\left(I + m(R+l)^2\right)\dot{\varphi}^2 + \frac{1}{2} m\,\dot{l}^2 - \frac{k}{2}(l - l_0)^2$$

Work out the canonical momenta and the Hamiltonian. Also, obtain the canonical equations of motion.

Solution

From the given Lagrangian,

$$p_l = \frac{\partial L}{\partial \dot{l}} = m\dot{l}, \qquad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \left(I + m(R+l)^2\right)\,\dot{\varphi}$$

The Hamiltonian is given by

$$\begin{aligned} H &= p_l\dot{l} + p_\varphi\dot{\varphi} - L = p_l\frac{p_l}{m} + p_\varphi\frac{p_\varphi}{(I + m(R+l)^2)} - L \\ &= p_l\frac{p_l}{m} + p_\varphi\frac{p_\varphi}{(I + m(R+l)^2)} - \frac{p_l^2}{2m} - \frac{p_\varphi^2}{2\,(I + m(R+l)^2)} + \frac{k}{2}(l - l_0)^2 \\ &= \frac{p_l^2}{2m} + \frac{p_\varphi^2}{2\,(I + m(R+l)^2)} + \frac{k}{2}(l - l_0)^2 \end{aligned}$$

As required, we eliminated velocities in H by solving for them in terms of the canonical momenta. The canonical equations of motion are:

$$\begin{aligned} \dot{l} &= \frac{\partial H}{p_l} = \frac{p_l}{m}, \qquad \dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{(I + m(R+l)^2)} \\ \dot{p}_l &= -\frac{\partial H}{\partial l} = -m(R+l)\frac{p_\varphi^2}{(I + m(R+l)^2)^2} \end{aligned}$$

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0$$

It is easy to check that these are equivalent to the Lagrangian equations of motion.

Problem 2 (10 points)

We consider a cylinder of height h, radius of cross section R, with a wedge of angle 2α cut out as shown. Obtain the moment of inertia of the object for rotations around the z-axis. (The cylinder is solid, I show the hollow picture to illustrate the coordinate system.)

Solution

We use the formula

$$I = I_{33} = \int d^3x\, \rho(x^2 + y^2)$$

for rotations around the z-axis. In cylindrical coordinates, this reduces to

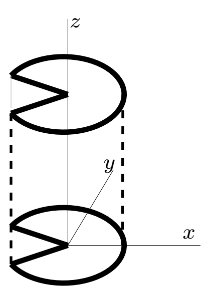
$$I = \int dz r dr d\varphi\, \rho\, r^2 = \int \rho\, r^3 dr\, dz d\varphi = \rho \frac{R^4}{4}\, h\, 2(\pi - \alpha)$$

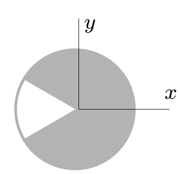
where we use the fact that the range of φ is from $-(\pi - \alpha)$ to $\pi - \alpha$ because of the wedge which has been cut out. In a similar way, the mass of the cylinder is given by

$$M = \int \rho\, r dr\, dz d\varphi = \rho \frac{R^2}{2}\, h\, 2(\pi - \alpha)$$

Comparing the two expressions, we see that the moment of inertia may be written as

$$I = \tfrac{1}{2} M R^2$$





Cross section of cylinder
with wedge cut out

Problem 2

Problem 3 (10 points)

The Lagrangian describing the motion of a particle of mass m in a frame which is rotating with angular velocity $\vec{\omega}$ was given in class as

$$L = \frac{1}{2} m \left(\dot{x}^2 - 2\, \epsilon_{ijk} \dot{x}^i x^j \omega^k + \omega^2 x^2 - (\vec{\omega} \cdot \vec{x})^2 \right) - V(x)$$

Write down the canonical momenta, the Hamiltonian and the Hamiltonian equations of motion.

Solution

a) In a straightforward way, we find the canonical momenta as

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}_i i - m\epsilon_{ijk}x^j\omega^k \implies \dot{x}_i = \frac{p_i + m\epsilon_{ijk}x^j\omega^k}{m}$$

The Hamiltonian is thus

$$H = p_i\dot{x}^i - L = \frac{(p_i + mA_i)^2}{2m} - \frac{m}{2}\left(\omega^2 x^2 - (\vec{\omega}\cdot\vec{x})^2\right) + V, \qquad A_i = \epsilon_{ijk}x^j\omega^k$$

The Hamiltonian equations of motion are

$$\begin{aligned} \dot{x}^i &= \frac{\partial H}{\partial p_i} = \frac{p_i + m\epsilon_{ijk}x^j\omega^k}{m} \\ \dot{p}_i &= -\frac{\partial H}{\partial x^i} = \epsilon_{ijk}(p + mA)^j\omega^k + m\left[\omega^2 x_i - \vec{\omega}\cdot\vec{x}\,\omega_i\right] - \frac{\partial V}{\partial x^i} \end{aligned}$$

Problem 4 (10 points)

For two particles of masses $m_1,\ m_2$ moving in one dimension, the transformation to the center of mass and relative coordinates is given by

$$P_1 = p_1 + p_2, \qquad Q_1 = \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2}$$
$$P_2 = \frac{m_2 p_1 - m_1 p_2}{m_1 + m_2}, \qquad Q_2 = q_1 - q_2$$

Show that this transformation is canonical. (*Hint: Write* $(P_1, Q_1, P_2, Q_2) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *and* $(p_1, q_1, p_2, q_2) = (\xi_1, \xi_2, \xi_3, \xi_4)$. *Then show that the Poisson bracket of two functions, say,* f *and* g*, is the same using the* λ*'s or the* ξ*'s.*

Solution

It is useful to have a compact notation. Write $(P_1, Q_1, P_2, Q_2) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(p_1, q_1, p_2, q_2) = (\xi_1, \xi_2, \xi_3, \xi_4)$. The given relations can be summarized as

$$\lambda_k = G_{ki}\xi_i, \qquad G = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha & 0 & \beta \\ \beta & 0 & -\alpha & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \qquad \alpha = \frac{m_1}{m_1 + m_2}, \quad \beta = \frac{m_2}{m_1 + m_2}$$

Notice that $\alpha + \beta = 1$. We will prove that the transformation is canonical by showing that the Poisson bracket of any two functions considered in terms of λ_k and in terms of ξ_i are the same. Define the matrix

$$\Omega^{-1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The Poisson bracket for f and h in terms of the ξ's is given as

$$\{f, h\}_\xi = (\Omega^{-1})^{ij} \frac{\partial f}{\partial \xi^i} \frac{\partial h}{\partial \xi^j}$$

Now assume that the arguments of the functions f and h are expressed in terms of the λ's. Then the expression above can be written by chain rule as

$$\begin{aligned} \{f, h\}_\xi &= (\Omega^{-1})^{ij} \frac{\partial \lambda_k}{\partial \xi^i} \frac{\partial \lambda_l}{\partial \xi^j} \frac{\partial f}{\partial \lambda^k} \frac{\partial h}{\partial \lambda^l} = (\Omega^{-1})^{ij} G_{ki} G_{lj} \frac{\partial f}{\partial \lambda^k} \frac{\partial h}{\partial \lambda^l} \\ &= (G \Omega^{-1} G^T)^{kl} \frac{\partial f}{\partial \lambda^k} \frac{\partial h}{\partial \lambda^l} \end{aligned}$$

In the last line we used a matrix notation for the product of the G's and Ω^{-1}. The required matrix product is

$$\begin{aligned} G\,\Omega^{-1} G^T &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha & 0 & \beta \\ \beta & 0 & -\alpha & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \beta & 0 \\ 0 & \alpha & 0 & 1 \\ 1 & 0 & -\alpha & 0 \\ 0 & \beta & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 & -1 \\ \alpha & 0 & \beta & 0 \\ 0 & -\beta & 0 & \alpha \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \beta & 0 \\ 0 & \alpha & 0 & 1 \\ 1 & 0 & -\alpha & 0 \\ 0 & \beta & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \Omega^{-1} \end{aligned}$$

Using this relation we see that

$$\{f, h\}_\xi = (\Omega^{-1})^{kl} \frac{\partial f}{\partial \lambda^k} \frac{\partial h}{\partial \lambda^l} = \{f, h\}_\lambda$$

This shows that the transformation to the center of mass and relative coordinates is a canonical transformation.