PHY 71100: ANALYTICAL DYNAMICS

Problem Set 7

Due December 9, 2024

Problem 1 (10 points)

Consider the motion of a particle of mass m moving freely, but confined to the surface of a sphere. Write down the Lagrangian, the Hamiltonian and then solve for the trajectories using the Hamilton-Jacobi method.

Solution

The distance between two points on the sphere with infinitesimal coordinate separation $(d\theta, d\varphi)$ is

$$ds^2 = R^2 \left[d\theta^2 + \sin^2\theta \, d\varphi^2\right]$$

Accordingly the Lagrangian for a particle moving freely on the surface of a sphere is

$$L = \tfrac{1}{2}\mu \left[\dot{\theta}^2 + \sin^2\theta \, \dot{\varphi}^2\right], \qquad \mu = mR^2$$

The canonical momenta are $p_\theta = \mu\dot{\theta}$, $p_\varphi = \mu \sin^2\theta \, \dot{\varphi}$. The Hamiltonian is thus given by

$$H = p_i \dot{q}_i - L = \frac{1}{2\mu}\left[p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right]$$

The H-J equation is thus given as

$$\frac{\partial S}{\partial t} + \frac{1}{2\mu}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{2\mu\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 = 0$$

We can separate variables using the substitution

$$S = W - Et + l\,\varphi$$

This leads to

$$W = \int d\theta \sqrt{2\mu E - \frac{l^2}{\sin^2\theta}}$$

Differentiating with respect to the constants of integration (E and l) and setting them to constants, we get

$$t - t_0 = \mu \int \frac{d\theta}{\sqrt{2\mu E - (l^2/\sin^2\theta)}} \tag{1}$$

$$\varphi - \varphi_0 = l \int \frac{d\theta}{\sin^2\theta\sqrt{2\mu E - (l^2/\sin^2\theta)}} \tag{2}$$

The first integral can be done easily using the substitution

$$\cos\theta = \sqrt{\frac{2\mu E - l^2}{2\mu E}}\,\cos\lambda$$

The result is

$$\cos\theta = \sqrt{\frac{2\mu E - l^2}{2\mu E}}\,\cos\left[\sqrt{\frac{2E}{\mu}}(t - t_0)\right] \tag{3}$$

For $l = 0$, these trace out longitudes connecting the north and south poles. Notice that, from (2), $\varphi = \varphi_0$ for $l = 0$. For $l \neq 0$, it is easier to take the time-derivative of (2) and substitute for $\dot\theta$ from (3). This gives

$$\dot\varphi = \frac{l}{\mu\sin^2\theta}$$

Using the expression for $\sin\theta$, this gives

$$\varphi - \varphi_0 = \beta\int\frac{d\lambda}{1 - (1 - \beta^2)\cos^2\lambda}, \qquad \lambda = \sqrt{\frac{2E}{\mu}}(t - t_0) \tag{4}$$

Problem 2 (15 points)

Consider a harmonic oscillator with two generalized coordinates and with the same frequency ω. The Hamiltonian is given by

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\omega^2}{2}(q_1^2 + q_2^2)$$

Defining $p_i = \sqrt{m\omega}P_i$, $q_i = q_i/\sqrt{m\omega}$, this becomes

$$H = \frac{\omega}{2}\left[P_1^2 + Q_1^2 + P_2^2 + Q_2^2\right]$$

The Poisson brackets for the new variables are the same as those for the old ones, since the scale factor will cancel between momenta and coordinates. Define three functions

$$K_1 = \frac{1}{2}\left[Q_1Q_2 + P_1P_2\right], \qquad K_2 = \frac{1}{2}\left[Q_1P_2 - Q_2P_1\right], \qquad K_3 = \frac{1}{4}\left[P_1^2 + Q_1^2 - P_2^2 - Q_2^2\right]$$

Calculate the Poisson brackets $\{K_i, K_j\}$ and $\{K_i, H\}$. What conclusion can you draw from these Poisson bracket relations?

Solution

The fundamental Poisson brackets are

$$\{Q_i, Q_j\} = 0, \qquad \{P_i, P_j\} = 0, \qquad \{Q_i, P_j\} = \delta_{ij}, \qquad i, j = 1, 2$$

Further, Poisson brackets obey the Leibniz rule

$$\{A, BC\} = \{A, B\}\, C + B\, \{A, C\}$$

Multiple use of this relation can be used to simplify the Poisson brackets for the K's.

$$\begin{aligned}
\{K_1, K_2\} &= \frac{1}{4}\{Q_1Q_2 + P_1P_2, Q_1P_2 - Q_2P_1\} \\
&= \frac{1}{4}\Big[Q_1^2\{Q_2, P_2\} - Q_2\{Q_1, P_1\}Q_2 + \{P_1, Q_1\}P_2^2 - P_1\{P_2, Q_2\}P_1\Big] \\
&= \frac{1}{4}\Big[Q_1^2 - Q_2^2 - P_2^2 + P_1^2\Big] = K_3
\end{aligned}$$

In going to second line of this equation, we used the Leibniz rule and kept only terms which could give a nonzero value. Thus terms like $\{Q_1, P_2\}Q_2P_2$ have been dropped since it is obviously zero from the fundamental brackets.

$$\begin{aligned}
\{K_2, K_3\} &= \frac{1}{8}\{Q_1P_2 - Q_2P_1, P_1^2 + Q_1^2 - P_2^2 - Q_2^2\} \\
&= \frac{1}{8}\Big[2\{Q_1, P_1\}P_1P_2 + Q_1(-2Q_2)\{P_2, Q_2\} - Q_2 2Q_1\{P_1, Q_1\} + 2\{Q_2, P_2\}P_2P_1\Big] \\
&= \frac{1}{2}(Q_1Q_2 + P_1P_2) = K_1
\end{aligned}$$

$$\begin{aligned}
\{K_3, K_1\} &= \frac{1}{8}\{P_1^2 + Q_1^2 - P_2^2 - Q_2^2, Q_1Q_2 + P_1P_2\} \\
&= \frac{1}{8}\Big[2P_1\{P_1, Q_1\}Q_2 + 2Q_1\{Q_1, P_1\}P_2 - 2Q_1P_2\{P_2, Q_2\} - 2Q_2P_1\{Q_2, P_2\}\Big] \\
&= \frac{1}{2}(Q_1P_2 - Q_2P_1) = K_2
\end{aligned}$$

These relations may be collected together as

$$\{K_i, K_j\} = \epsilon_{ijk}\, K_k$$

Notice that these are identical to the bracket relations for three-dimensional angular momentum, although this is a two-dimensional oscillator and hence it is not directly related to the 3d-angular momentum.

We can now calculate $\{K_i, H\}$.

$$\begin{aligned}
\{K_1, H\} &= \frac{\omega}{4}\{Q_1Q_2 + P_1P_2, P_1^2 + Q_1^2 + P_2^2 + Q_2^2\} \\
&= \frac{\omega}{4}\Big[2P_1\{Q_1, P_1\}Q_2 + 2Q_1P_2\{Q_2, P_2\} + 2\{P_1, Q_1\}P_2Q_1 + 2P_1Q_2\{P_2, Q_2\}\Big] \\
&= \frac{\omega}{4}\Big[2P_1Q_2 + 2Q_1P_2 - 2P_2Q_1 - 2P_1Q_2\Big] = 0
\end{aligned}$$

$$
\begin{aligned}
\{K_2, H\} &= \frac{\omega}{4}\{Q_1P_2 - Q_2P_1, P_1^2 + Q_1^2 + P_2^2 + Q_2^2\} \\
&= \frac{\omega}{4}\Big[2\{Q_1, P_1\}P_1P_2 + 2Q_1\{P_2, Q_2\}Q_2 - 2Q_2Q_1\{P_1, Q_1\} - 2\{Q_2, P_2\}P_1P_2\Big] \\
&= \frac{\omega}{4}\Big[2P_1P_2 - 2Q_1Q_2 + 2Q_2Q_1 - 2P_1P_2\Big] = 0
\end{aligned}
$$

Since $\{K_1, K_2\} = K_3$,

$$
\begin{aligned}
\{K_3, H\} &= \{\{K_1, K_2\}, H\} \\
&= -\{\{H, K_1\}, K_2\} - \{\{K_2, H\}, K_1\} = 0
\end{aligned}
$$

where we used the Jacobi identity in the second line,

$$
\{\{A, B\}, C\} + \{\{C, A\}, B\} + \{\{B, C\}, A\} = 0
$$

We have shown that

$$
\{K_i, H\} = 0, \qquad \{K_i, K_j\} = \epsilon_{ijk}K_k
$$

Since K_i have zero PB with H, the transformations generated by the K_i are a symmetry of the problem; i.e., they leave the Hamiltonian unchanged. Further since

$$
\frac{\partial K_i}{\partial t} = \{K_i, H\}
$$

we see that K_i are also conserved. We have shown that there is hidden symmetry for the harmonic oscillator with two generalized coordinates for the same frequency for both and that this leads, via Noether's theorem, to conservation laws.

Problem 3 (10 points)

Consider the Lorentz transformation of a vector given in matrix notation as $x' = Lx$ where x, x' are column vectors with 4 entries each,

$$
x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \qquad x' = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}
$$

and L denotes the 4×4 Lorentz transformation matrix. For velocity along the x^1-direction I obtained L in class as

$$
L = L^{(1)} = \begin{bmatrix} \gamma & -\gamma\, v_1/c & 0 & 0 \\ -\gamma\, v_1/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where $\gamma = \frac{1}{\sqrt{1-v_1^2/c^2}}$.

a) Write down the matrix $L^{(2)}$ for transformation along the x^2-direction with velocity v_2.

b) Calculate the commutator $L^{(1)}L^{(2)} - L^{(2)}L^{(1)}$. Simplify the commutator for small velocities, keeping $v_1 v_2$ term, but not higher powers of v's. Can you identify the resulting transformation? (*This is related to the Thomas precession which helps to identify the correct spin magnetic moment.*)

Solution

The matrix $L^{(1)}$ is given as

$$L^{(1)} = \begin{bmatrix} \gamma & -\gamma\, v_1/c & 0 & 0 \\ -\gamma\, v_1/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-v_1^2/c^2}}$$

$L^{(2)}$ is similar with x_2 taking the place of x_1, so we get

$$L^{(2)} = \begin{bmatrix} \gamma' & 0 & -\gamma'\, v_2/c & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma'\, v_2/c & 0 & \gamma' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \gamma' = \frac{1}{\sqrt{1-v_2^2/c^2}}$$

By straightforward multiplication, we find

$$L^{(1)}L^{(2)} = \begin{bmatrix} \gamma\gamma' & -\gamma v_1/c & -\gamma\gamma' v_2/c & 0 \\ -\gamma\gamma' v_1/c & \gamma & \gamma\gamma' v_1 v_2/c^2 & 0 \\ -\gamma' v_2/c & 0 & \gamma' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$L^{(2)}L^{(1)} = \begin{bmatrix} \gamma\gamma' & -\gamma\gamma' v_1/c & -\gamma' v_2/c & 0 \\ -\gamma v_1/c & \gamma & 0 & 0 \\ -\gamma\gamma' v_2/c & \gamma\gamma' v_1 v_2/c^2 & \gamma' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Subtracting the second from the first,

$$L^{(1)}L^{(2)} - L^{(2)}L^{(1)} = \begin{bmatrix} 0 & (\gamma'-1)\gamma v_1/c & -(\gamma-1)\gamma' v_2/c & 0 \\ -(\gamma'-1)\gamma v_1/c & 0 & \gamma\gamma' v_1 v_2/c^2 & 0 \\ (\gamma-1)\gamma' v_2/c & -\gamma\gamma' v_1 v_2/c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly Lorentz transformations along different directions do not commute. For small velocities, $\gamma \approx 1 + \mathcal{O}(v^2/c^2)$. In this approximation, we find

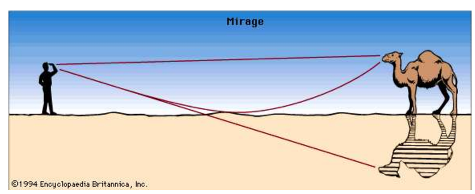
$$L^{(1)}L^{(2)} - L^{(2)}L^{(1)} = \frac{v_1 v_2}{c^2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that the matrix on the right hand side is basically the rotation generator J_3 theta we have discussed before. We see that the commutator of the two Lorentz transformations is a spatial rotation by an angle $\theta_3 = v_1 v_2/c^2$, for small velocities.

Problem 4 (10 points)

Considering light as made of photons, one can use the relation between energy and momentum of a photon to describe the propagation of light. This relation is given by $p_0^2 - c^2 \vec{p} \cdot \vec{p} = 0$, where p_0 is the Hamiltonian. Replacing momenta by derivatives of the action S, namely, using

$$p_0 = H = -\frac{\partial S}{\partial t}, \qquad p_i = \frac{\partial S}{\partial x^i},$$



we get the corresponding Hamilton-Jacobi equation

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \nabla S \cdot \nabla S = 0$$

In a medium with an index of refraction $n(x)$, this is modified to

$$\frac{n^2}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \nabla S \cdot \nabla S = 0$$

On a hot day on a tarred road or in a desert, the index n is lower near the surface and increases to some higher value as we go up, because of the higher temperatures closer to the ground. This makes light rays going down bend back upwards creating the illusion of seeing upside down images as if they are reflected in a pool of water. This is the phenomenon of mirage. We can apply the H-J equation to analyze this. Consider modeling the index of refraction as

$$n^2(x, y) = n_0^2 - b^2 e^{-2ay}, \qquad y \geq 0$$

Here x is the horizontal direction, y denotes the vertical direction. Solve and identify the trajectories of the photons or light rays with this index of refraction.

Solution

The Hamilton-Jacobi equation has been given as

$$\frac{n^2}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \nabla S \cdot \nabla S = 0$$

Since n is independent of x, we can separate variables in the H-J equation by

$$S = W(y) - \omega t + kx$$

This leads to

$$W = \frac{\omega}{c}\int dy\,\sqrt{n^2 - \alpha^2}, \qquad \alpha = \frac{ck}{\omega}$$

We differentiate S with respect to the constant of integration k and set it to x_0 to obtain

$$x - x_0 = \alpha \int \frac{dy}{\sqrt{n^2 - \alpha^2}} = \alpha \int \frac{dy}{\sqrt{n_0^2 - \alpha^2 - b^2 e^{-2ay}}}$$

Introduce the variable $u = e^{ay}$. We write the integral as

$$\begin{aligned} I &= \int \frac{dy}{\sqrt{n_0^2 - \alpha^2 - b^2 e^{-2ay}}} = \int \frac{dy\, e^{ay}}{\sqrt{(n_0^2 - \alpha^2)e^{2ay} - b^2}} \\ &= \frac{1}{a}\int \frac{du}{\sqrt{(n_0^2 - \alpha^2)u^2 - b^2}} \\ &= \frac{1}{a\sqrt{n_0^2 - \alpha^2}} \cosh^{-1}\left[\frac{\sqrt{n_0^2 - \alpha^2}\,u}{b}\right] \end{aligned}$$

Using this result, we get the trajectory as

$$e^{ay} = \frac{b}{\sqrt{n_0^2 - \alpha^2}} \cosh\left[\frac{a\sqrt{n_0^2 - \alpha^2}}{\alpha}(x - x_0)\right]$$

Caution: x_0 is not the initial value of x. Notice that, at x_0, y has its minimum value. So x_0 corresponds to the lowest point of the trajectory.