PHY 71100: ANALYTICAL DYNAMICS

Problem Set 7

Due December 9, 2024

Problem 1

Consider the motion of a particle of mass m moving freely, but confined to the surface of a sphere. Write down the Lagrangian, the Hamiltonian and then solve for the trajectories using the Hamilton-Jacobi method.

Problem 2

Consider a harmonic oscillator with two generalized coordinates and with the same frequency ω. The Hamiltonian is given by

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\omega^2}{2}(q_1^2 + q_2^2)$$

Defining $p_i = \sqrt{m\omega}P_i$, $q_i = q_i/\sqrt{m\omega}$, this becomes

$$H = \frac{\omega}{2}\left[P_1^2 + Q_1^2 + P_2^2 + Q_2^2\right]$$

The Poisson brackets for the new variables are the same as those for the old ones, since the scale factor will cancel between momenta and coordinates. Define three functions

$$K_1 = \frac{1}{2}\left[Q_1 Q_2 + P_1 P_2\right], \qquad K_2 = \frac{1}{2}\left[Q_1 P_2 - Q_2 P_1\right], \qquad K_3 = \frac{1}{4}\left[P_1^2 + Q_1^2 - P_2^2 - Q_2^2\right]$$

Calculate the Poisson brackets $\{K_i, K_j\}$ and $\{K_i, H\}$. What conclusion can you draw from these Poisson bracket relations?

Problem 3

Consider the Lorentz transformation of a vector given in matrix notation as $x' = Lx$ where x, x' are column vectors with 4 entries each,

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \qquad x' = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}$$

and L denotes the 4×4 Lorentz transformation matrix. For velocity along the x^1-direction I obtained L in class as

$$L = L^{(1)} = \begin{bmatrix} \gamma & -\gamma\, v_1/c & 0 & 0 \\ -\gamma\, v_1/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

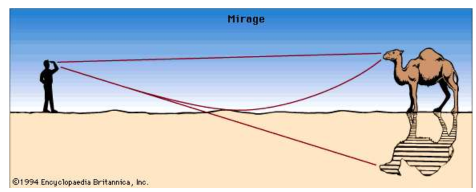
where $\gamma = \frac{1}{\sqrt{1-v_1^2/c^2}}$.

a) Write down the matrix $L^{(2)}$ for transformation along the x^2-direction with velocity v_2.

b) Calculate the commutator $L^{(1)}L^{(2)} - L^{(2)}L^{(1)}$. Simplify the commutator for small velocities, keeping $v_1 v_2$ term, but not higher powers of v's. Can you identify the resulting transformation? (*This is related to the Thomas precession which helps to identify the correct spin magnetic moment.*)

Problem 4

Considering light as made of photons, one can use the relation between energy and momentum of a photon to describe the propagation of light. This relation is given by $p_0^2 - c^2 \vec{p} \cdot \vec{p} = 0$, where p_0 is the Hamiltonian. Replacing momenta by derivatives of the action S, namely, using



$$p_0 = H = -\frac{\partial S}{\partial t}, \qquad p_i = \frac{\partial S}{\partial x^i},$$

we get the corresponding Hamilton-Jacobi equation

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \nabla S \cdot \nabla S = 0$$

In a medium with an index of refraction $n(x)$, this is modified to

$$\frac{n^2}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \nabla S \cdot \nabla S = 0$$

On a hot day on a tarred road or in a desert, the index n is lower near the surface and increases to some higher value as we go up, because of the higher temperatures closer to the ground. This makes light rays going down bend back upwards creating the illusion of seeing upside down images as if they are reflected in a pool of water. This is the phenomenon of mirage. We can apply the H-J equation to analyze this. Consider modeling the index of refraction as

$$n^2(x, y) = n_0^2 - b^2 e^{-2ay}, \qquad y \geq 0$$

Here x is the horizontal direction, y denotes the vertical direction. Solve and identify the trajectories of the photons or light rays with this index of refraction.