1. Isometries, Poincaré Algebra

1.1 Isometries and the Killing equation

We start with a description of how we can build up physical theories based on some elemental concepts. We will take the notion of the spacetime manifold as foundational to physical theories. From everything we know so far, most of the physics over length scales 10^{-18} m to 10^{15} m plays out on a continuous differentiable spacetime manifold. This may change for physics at shorter distances when effects of quantum gravity are taken into account. One may need something like noncommutative geometry or related concepts. But for what we want to do spacetime manifold is good starting point.

A manifold is defined as follows. We start with a topological space which means that we have a set of elements (points) with a definition of open sets or open neighborhoods. Manifold We also have continuous and invertible maps from such sets to open sets in $\mathbb{R}^n$, for some integer n (which is the dimension of the space). The values in $\mathbb{R}^n$ are the coordinates of the point which is its pre-image on the topological space. In short, basically, we use coordinates to describe points in the space of interest. Such coordinates should be single-valued by construction since the maps from the space to $\mathbb{R}^n$ are invertible. This can mean that there is no globally defined coordinate system for the whole space; we have to use different systems for different neighborhoods. For a differentiable manifold, the maps from the space to $\mathbb{R}^n$ are differentiable.

Starting with a differentiable manifold, one can define vectors, differential forms, etc.; these will turn out to be important for many questions which we will discuss later on. But for now, we will consider manifolds with an additional structure on them, namely, the notion of a metric.

The metric is the measure of distance in any given physical coordinate system. In Metric Newtonian mechanics, we are familiar with the distance between two nearby points, say ds, playing a central role in its formulation. For example, for two points which are infinitesimally close to each other, the coordinates in the Cartesian system may be taken as (x, y, z) and

$(x + dx, y + dy, z + dz)$, The distance between these two points is given by the Pythagorean theorem as

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{1.1}$$

We can also measure the distance in other coordinate systems by appropriate transformation of coordinates. For example, to use cylindrical coordinates, the transformation of coordinates is

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \tag{1.2}$$

The metric (1.1) expressed in the cylindrical coordinates becomes

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 = (d\,(r\cos\theta))^2 + (d\,(r\sin\theta))^2 + dz^2 \\ &= dr^2 + r^2 d\theta^2 + dz^2 \end{aligned} \tag{1.3}$$

In a similar way, one can introduce spherical polar coordinates by the transformation

$$x = r\,\sin\theta\,\cos\varphi, \quad y = r\,\sin\theta\,\sin\varphi, \quad z = r\,\cos\theta \tag{1.4}$$

with the corresponding expression for the metric

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2\,\sin^2\theta\,d\varphi^2 \tag{1.5}$$

More generally, we consider the metric to be of the form

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu \tag{1.6}$$

where $\mu,\ \nu = 1, 2, \cdots, n$ and we use the summation convention. Any repeated index implies summation over all values of the index. Thus in (1.6), we have summation over μ and ν from 1 to n. $g_{\mu\nu}$ can in general be functions of the coordinates and is referred to as the metric tensor. If we change coordinates from x^μ to y^μ, we see that we can take x^μ to be functions of the y-coordinates so that

$$dx^\mu = \frac{\partial x^\mu}{\partial y^\alpha} dy^\alpha \tag{1.7}$$

Thus, (**??**) becomes

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu = g_{\mu\nu}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}dy^\alpha dy^\beta = g_{\alpha\beta}(y)dy^\alpha dy^\beta \tag{1.8}$$

This identifies the components of the metric tensor in the y-system as

$$g_{\alpha\beta}\big]_y = g_{\mu\nu}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta} \tag{1.9}$$

We now turn to some issues specific to physics. The basic recognition which emerged from Einstein's special theory of relativity (and which was reinforced by the general theory of relativity) is that the world is best described as a four-dimensional spacetime manifold, where we have the usual 3 spatial coordinates and time which is treated as the fourth (or zeroth) coordinate. Thus it is the *spacetime* metric, where space and time are treated on a roughly equal footing, that is relevant for physics. The key entities of interest are "events" which take place at a given point in space (with coordinates x^1, x^2, x^3) and at a given time (with time-coordinate x^0). In general, the metric is determined by the physical matter content via the Einstein field equations for gravity. We will postpone the discussion of this equation for now; the metric which will play an important role for what we want to do now is the Minkowski metric or Minkowski spacetime which is the special case with zero curvature. This metric is given by

Minkowski metric

$$\begin{aligned} ds^2 &= c^2 dt^2 - (dx^1)^2 + (dx^2)^2 + (dx^3)^2) = \eta_{\mu\nu} dx^\mu dx^\nu \\ \eta_{00} &= -\eta_{11} = -\eta_{22} = -\eta_{33} = 1, \quad \eta_{0i} = \eta_{ij} = 0 \quad i \neq j \end{aligned} \tag{1.10}$$

where c is the speed of light in vacuum. Notice the relative sign difference between the temporal part and the spatial parts; as a result ds^2 need not have a definite sign anymore. (The overall sign in (1.10) is a matter of convention; a different convention, which is equally acceptable, would be to choose the spatial parts positive and the temporal part negative.) Hereafter we will set $c = 1$ which can be done by a proper choice of units, so that we write $x^0 = t$. Equation (1.10) is given in a Cartesian coordinate system; if needed one can make coordinate changes. We may say that Minkowski spacetime corresponds to the special case of the metric tensor $g_{\mu\nu} = \eta_{\mu\nu}$. Sometimes it is useful to write the metric tensor as a 4×4-matrix,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.11}$$

Notice that the metric tensor is always an invertible matrix. (If not, the spacetime is singular.) For the Minkowski case, the inverse is written with superscripts, i.e., $(\eta^{-1})^{\mu\nu} = \eta^{\mu\nu}$, and obeys

$$\eta^{\mu\nu} \eta_{\nu\alpha} = \delta^\mu_\alpha \tag{1.12}$$

Also similar to the coordinate differentials dx^μ, we can introduce 4-vectors specified by components of the form A^μ, $\mu = 0, 1, 2, 3$. The scalar product of two such vectors with components A^μ, B^ν can be expressed using the metric tensor as

$$A \cdot B = \eta_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \vec{A} \cdot \vec{B} \tag{1.13}$$

We also define vectors (or covectors) with lower indices by $A_\mu = \eta_{\mu\nu}A^\nu$, so that we may write the scalar product as $A \cdot B = A_\mu B^\mu$.

We are interested in symmetries of the metric, for reasons which will become clear when we introduce the principle of relativity. Symmetries of the metric, namely transformations which preserve the metric, are referred to as isometries. There are two types of isometries of interest, discrete and continuous. Discrete isometries of the metric are easy to identify.

Time-reversal

Notice that the metric (1.10) is unchanged under

$$x^0 \to -x^0 \tag{1.14}$$

Parity

This is known as time-reversal symmetry. Another isometry is

$$x^i \to -x^i, \qquad i = 1, 2, 3 \tag{1.15}$$

This is parity or reflection of coordinates.

We now turn to the continuous isometries. (We will write down the required condition for an isometry for a general metric before specializing to the Minkowski case.) A change of coordinates will leave the metric invariant by construction, since the metric tensor is taken to transform as in (). Here we are still considering the metric at the same point which may be described either in terms of x^μ or in terms of y^μ. This is not what is meant by an isometry. For a continuous isometry we consider a nearby point with coordinates $x^\mu + \xi^\mu(x)$, where $\xi^\mu(x)$ is a suitably chosen function of x^μ such that the metric defined around $x^\mu + \xi^\mu$ is the same as the one defined at x^μ. For a continuous isometry, we can take ξ^μ to be small or infinitesimal to carry out the analysis. The condition that this change is an isometry is thus

$$(ds^2)_{x+\xi} = (ds^2)_x \tag{1.16}$$

Written out, we find

$$\begin{aligned} (ds^2)_{x+\xi} - (ds^2)_x &= g_{\mu\nu}(x+\xi)\, d(x+\xi)^\mu\, d(x+\xi)^\nu - g_{\mu\nu}(x) dx^\mu dx^\nu \\ &= \left[\xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\alpha\nu}\frac{\partial \xi^\alpha}{\partial x^\mu} + g_{\mu\alpha}\frac{\partial \xi^\alpha}{\partial x^\nu} \right] dx^\mu dx^\nu \end{aligned} \tag{1.17}$$

We have expanded the quantity on the right hand side to first order in ξ^μ, in accordance with the idea of treating it as infinitesimal. From (1.17), we see that the transformation

Killing equation

$x^\mu \to x^\mu + \xi^\mu$ is an isometry if ξ^μ obeys the equation

$$\xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\alpha\nu}\frac{\partial \xi^\alpha}{\partial x^\mu} + g_{\mu\alpha}\frac{\partial \xi^\alpha}{\partial x^\nu} = 0 \tag{1.18}$$

This is known as the Killing equation.

For the *cognoscenti*, this equation may seem a bit strange since it does not look covariant, involving just ordinary derivatives of ξ^α and $g_{\mu\nu}$. But using the definition of the Christoffel symbol,

$$\Gamma^\alpha_{\mu\lambda} = \frac{1}{2} g^{\alpha\beta} \left(-\frac{\partial g_{\mu\lambda}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\lambda} + \frac{\partial g_{\lambda\beta}}{\partial x^\mu} \right), \tag{1.19}$$

(1.18) can be written as

$$g_{\mu\alpha}\nabla_\nu\xi^\alpha + g_{\nu\alpha}\nabla_\mu\xi^\alpha = 0 \tag{1.20}$$

in terms of the covariant derivative

$$\nabla_\mu\xi^\alpha = \frac{\partial\xi^\alpha}{\partial x^\mu} + \Gamma^\alpha_{\mu\lambda}\xi^\lambda \tag{1.21}$$

Thus the Killing equation is indeed covariant under coordinate transformations.

1.2 Isometries of Minkowski space, Poincaré algebra

We now consider the Killing equation for the Minkowski metric. Since $g_{\mu\nu} = \eta_{\mu\nu}$ for this case, the equation becomes

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = 0 \tag{1.22}$$

Writing an ansatz for ξ_μ as a power series of the form

$$\xi_\mu = a_\mu + \omega_{\mu\nu}x^\nu + \frac{1}{2}\omega_{\mu\nu\alpha}x^\nu x^\alpha + \frac{1}{3!}\omega_{\mu\nu\alpha\beta}x^\nu x^\alpha x^\beta + \cdots \tag{1.23}$$

where the coefficients a_μ, $\omega_{\mu\nu}$, $\omega_{\mu\nu\alpha}$, etc. are constants, we find that (1.22) is satisfied if

$$\begin{aligned} \omega_{\mu\nu} &= -\omega_{\nu\mu} \\ \omega_{\mu\nu\alpha} &= -\omega_{\nu\mu\alpha}, \quad \text{etc.} \end{aligned} \tag{1.24}$$

Notice that, from (1.23), $\omega_{\mu\nu\alpha}$ is symmetric in the last two indices but antisymmetric in the first two as a result of (1.24). Thus

$$\begin{aligned} \omega_{\mu\nu\alpha} &= \omega_{\mu\alpha\nu} = -\omega_{\alpha\mu\nu} = -\omega_{\alpha\nu\mu} = \omega_{\nu\alpha\mu} = \omega_{\nu\mu\alpha} \\ &= -\omega_{\mu\nu\alpha} \quad \text{from (1.24)} \end{aligned} \tag{1.25}$$

Thus $\omega_{\mu\nu\alpha} = 0$. By a similar reasoning all higher terms in (1.24) are zero. The complete solution of the Killing equation for the Minkowski metric is thus given by

$$\xi_\mu = a_\mu + \omega_{\mu\nu}x^\nu, \quad \omega_{\mu\nu} = -\omega_{\nu\mu} \tag{1.26}$$

The parameters a_μ correspond to translations, while $\omega_{\mu\nu}x^\nu$, we will show presently, correspond to Lorentz boosts (connecting frames of reference moving at constant relative velocity) and spatial rotations. The Lorentz boosts together with spatial rotations constitute what is called the Lorentz group. Translations along with all Lorentz transformations (including rotations) form what is called the Poincaré group.

Since $\omega_{\mu\nu}$ is antisymmetric, there are six independent parameters ω_{01}, ω_{02}, ω_{03}, ω_{12}, ω_{23}, ω_{31}. Consider first the case of nonzero $\omega_{01} = \omega$, with all other $\omega_{\mu\nu} = 0$. The infinitesimal transformation $(x^\mu)' = x^\mu + \xi^\mu$ is given by

$$\begin{aligned}(x^0)' &= x^0 + \omega^0{}_1\, x^1 = x^0 + \omega\, x^1 \\ (x^1)' &= x^1 + \omega^1{}_0\, x^0 = x^1 + \omega\, x^0 \\ (x^2)' &= x^2, \qquad (x^3)' = x^3\end{aligned} \tag{1.27}$$

We can collect these as

$$\begin{aligned}\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}' &= \begin{bmatrix} 1 & \omega & 0 & 0 \\ \omega & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \\ &= \left(\mathbb{1} + \omega\, K_1\right) \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \qquad K_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned} \tag{1.28}$$

Since ω is infinitesimal, we can build up a finite transformation by using several such transformations in sequence. Let Ω be the finite value of the parameter corresponding to ω_{01}. We divide this into N units of ω each, $\Omega = N\omega$ so that we have N infinitesimal; transformations. The change in the x^μ is thus given by

$$(x^\mu)' = \left[(\mathbb{1} + \omega)^N\right]^\mu{}_\nu x^\nu = \left[\left(\mathbb{1} + \frac{\Omega}{N}\right)^N\right]^\mu{}_\nu x^\nu \tag{1.29}$$

The errors of order ω^2 disappear if we take the limit $N \to \infty$, $\omega \to 0$, keeping Ω finite. Using $\lim_{N\to\infty}(1 + X/N)^N = e^X$, we find

$$(x^\mu)' = L^\mu{}_\nu x^\nu, \qquad L = \exp(\Omega K_1) \tag{1.30}$$

Notice that

$$(K_1)^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (K_1)^3 = K_1, \qquad \text{etc.,} \tag{1.31}$$

we find, by expansion of the exponential,

$$L = \begin{bmatrix} \cosh\Omega & \sinh\Omega & 0 & 0 \\ \sinh\Omega & \cosh\Omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.32}$$

We now define a parameter v related to Ω as $\tanh\Omega = v$, so that

$$\cosh\Omega = \frac{1}{\sqrt{1-v^2}}, \qquad \sinh\Omega = \frac{v}{\sqrt{1-v^2}}, \tag{1.33}$$

The transformation (1.30) takes the form

$$\begin{aligned}
(x^0)' &= \frac{x^0 + vx^1}{\sqrt{1-v^2}} \\
(x^1)' &= \frac{x^1 + vx^0}{\sqrt{1-v^2}} \\
(x^2)' &= x^2, \qquad (x^3)' = x^3
\end{aligned} \tag{1.34}$$

We see that ω_{01} indeed corresponds to a Lorentz boost transformation along the x^1-axis, with velocity v. Similarly, ω_{02}, ω_{03} correspond to Lorentz transformations along the x^2- and x^3-axes.

Turning to the remaining ω's, consider $\omega_{12} \equiv \theta$. In this case, x^0 and x^3 are unchanged while

$$(x^1)' = x^1 + \omega^1{}_2 x^2 = x^1 - \theta x^2, \qquad (x^2)' = x^2 + \omega^2{}_1 x^2 = x^2 + \theta x^1 \tag{1.35}$$

The relevant matrix that takes the place of K_1 is now

$$J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.36}$$

Exponentiating as before and working out $(J_3)^2$, $(J_3)^3$, etc., we find that this corresponds to a rotation around the x^3-axis given by

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta & 0 \\ 0 & \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{1.37}$$

The other two parameters ω_{23} and ω_{31} give similar rotations around the x^1- and x^2-axes.

These are all transformations connected to the identity, since we start from the identity and we have a sequence of transformations connecting the finite transformation to the identity. Notice that in all these cases the determinant of the transformation matrix is 1. For the parity and time-reversal transformations, the determinant is -1 and cannot be obtained by infinitesimal deformations from identity. They are discrete and disconnected from the identity. However, we can compose parity and time-reversal with any of the continuous and connected transformations given above to get all the disconnected ones with $\det = -1$.

1.3 Unitary representations of the Poincaré algebra

We now have all the isometries of the Minkowski metric. They consist of all Poincaré transformations (translations in space , translations in time, three spatial rotations, three Lorentz boosts along the three axes) and the discrete transformations parity and time-reversal. So before we proceed, it si useful to pause and ask why thus is all relevant. The key idea here is what we shall refer to as the principle of relativity, although it may not look like the postulates of special relativity given in most books. This principle, which is foundational to all physics, is the following.

Proposition 1.1 — The principle of special relativity. In situations where gravity can be neglected or is not important, physics is invariant under the continuous isometries of the Minkowski metric $ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$ which are also connected to the identity.

The statement is approximate in the sense that gravitating bodies modify the metric (1.10). So the principle has to be suitably modified when gravity is included. But for most situations where gravitational effects are negligible, we can use the principle as stated. Further, discrete symmetries are not respected by all physics. Weak interactions break both parity and time-reversal symmetry.

We now turn to the algebraic characterization of these symmetries. This is important if we consider the question of how they act on the wave functions or the Hilbert space of states when we consider the quantum theory. From the definition of $(x^\mu)' = x^\mu + \xi^\mu$, we see that for infinitesimal transformations, we can write

$$\begin{aligned} f(x+\xi) &\approx f(x) + \xi^\mu \frac{\partial f}{\partial x^\mu} + \cdots \\ &\approx f(x) - i a^\mu P_\mu f + \frac{i}{2}\omega^{\mu\nu}(x_\mu P_\nu - x_\nu P_\mu) f \\ P_\mu &= i\frac{\partial}{\partial x^\mu} \end{aligned} \tag{1.38}$$

This identifies the translation generators as P_μ and the generators of the Lorentz transformations as

$$M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu \tag{1.39}$$

Poincaré algebra

The commutation algebra among these can be easily worked out. We get

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [M_{\mu\nu}, P_\alpha] &= -i\,(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu) \\ [M_{\mu\nu}, M_{\alpha\beta}] &= -i\left(\eta_{\mu\alpha}M_{\nu\beta} - \eta_{\mu\beta}M_{\nu\alpha} + \eta_{\nu\beta}M_{\mu\alpha} - \eta_{\nu\alpha}M_{\mu\beta}\right) \end{aligned} \tag{1.40}$$

This is known as the Poincaré algebra. From the definition of P_μ, we see that P_0 corresponding to $i\partial/\partial x^0$ denotes the energy operator in the quantum theory and P_i denote the momentum. Similarly, $J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$ denote the orbital angular momentum (generating rotations) and $K_i = M_{0i}$ generate Lorentz boosts. Given the principle of relativity, every physical theory (modulo the weak gravity caveat mentioned above) must have a realization of this algebra.

We considered the change in a function f. But if we consider the change in a vector A_ν, there is an additional change due to the change of the components under a Lorentz transformation as in (1.30). This gives

$$\begin{aligned}\delta A_\nu &= \frac{i}{2}\omega^{\alpha\beta}\left[\left(x_\alpha P_\beta - x_\beta P_\alpha\right) A_\nu + \left(-i\eta_{\nu\alpha}\delta^\lambda_\beta + i\eta_{\nu\beta}\delta^\lambda_\alpha\right) A_\lambda\right] \\ &= \frac{i}{2}\omega^{\alpha\beta}\left(M_{\alpha\beta}\right)^{\lambda}_{\nu} A_\lambda \end{aligned} \tag{1.41}$$

where the generator has a differential operator part and an internal transformation matrix (denoting spin) acting on the components; i.e.,

$$\begin{aligned}\left(M_{\alpha\beta}\right)^{\lambda}_{\nu} &= \left(x_\alpha P_\beta - x_\beta P_\alpha\right)\delta^\lambda_\nu + \left(S_{\alpha\beta}\right)^{\lambda}_{\nu} \\ \left(S_{\alpha\beta}\right)^{\lambda}_{\nu} &= -i\left(\eta_{\nu\alpha}\delta^\lambda_\beta - \eta_{\nu\beta}\delta^\lambda_\alpha\right)\end{aligned} \tag{1.42}$$

Treating ν, λ as matrix labels and with matrix multiplication of the S's, it is easy to see that they obey similar commutation rules as in (1.40). In other words, the Poincaré algebra (1.40) is obtained even when vector fields or tensor fields (or even spinor fields although we have not talked about them yet) are included.

A physical theory in general will have many more operators of interest than just P_μ, $M_{\mu\nu}$. Let us denote the algebra of all the observables of a theory by $\mathcal{A}$. Then, what we are saying is that this must contain the Poincaré algebra, i.e., Poincaré algebra $\subset \mathcal{A}$. The minimal case would be a theory where the Poincaré algebra gives all the operators of interest. We will analyze this situation now and show that this corresponds to what we may consider as point particles.

Towards this, recall that the quantum mechanics of a physical system may be defined as a unitary irreducible representation of the algebra of observables. The requirement of unitarity is clear since all physical observables generate unitary transformations on the Hilbert space in quantum mechanics. Irreducibility is less obvious, so a comment on this will be useful at this stage. Let us consider nonrelativistic quantum mechanics and just one-particle dynamics in one dimension. The algebra is given by the position and momentum operators, $\hat{x}$ and $\hat{p}$, with the commutation rules

$$[\hat{x}, \hat{x}] = 0, \quad [\hat{p}, \hat{p}] = 0, \quad [\hat{x}, \hat{p}] = i \tag{1.43}$$

We need a unitary representation of this algebra of observables. One way to represent the operators $\hat{x}, \hat{p}$ is as follows. We can consider square-integrable complex-valued functions $\psi(x, p)$ and take the action of the operators as

$$\hat{p}\psi = -i\frac{\partial}{\partial x}\psi$$

$$\hat{x}\,\psi \;=\; \left(i\frac{\partial}{\partial p}+x\right)\psi \tag{1.44}$$

This is consistent with the Heisenberg algebra (1.43) but it is not right since we are specifying the wave functions as functions of x and p. For example, when we write down the Schrödinger equation, what value do we pick for p? Different values will give different physical results. We can also see that this representation is reducible. For this, consider imposing the condition

$$\frac{\partial \psi}{\partial p}=0 \tag{1.45}$$

on the wave functions and still obtain a representation of (1.43). In particular, in this case,

$$\hat{p}\,\psi \;=-i\frac{\partial}{\partial x}\psi, \qquad \hat{x}\,\psi \;=x\,\psi \tag{1.46}$$

This is the usual Schrödinger representation. Since we are able to obtain a representation on this smaller space of functions obeying the condition (1.45), the former representation (1.44) is reducible. The Schrödinger representation is irreducible; i.e., there is no “smaller" function space on which the Heisenberg algebra (1.43) can be realized. (Properly speaking, one should consider bounded operators obtained by exponentiation.)

We see that unitarity and irreducibility of the algebra of observables are the features needed to define quantum mechanics. (For the algebra (1.43), there is only one representation, the Schrödinger representation, up to unitary equivalence. This is a theorem due to Stone and von Neumann. If we have an infinite number of degrees of freedom, i.e., p_i and x_i where $i=1,2,\cdots,\infty$, or if the space is not simply connected, there can be many UIR’s. These are realized for phase transitions in quantum field theory and for fractional spin in the case of quantum Hall effect, etc. We will discuss these later.

Returning to the present case, we will consider realization of the algebra (1.40) where the transformations $e^{ia^\mu P_\mu}$ and $e^{\frac{i}{2}\omega^{\alpha\beta}M_{\alpha\beta}}$ are unitary operators. Notice that this is not so trivial; for example, the 4×4 transformation matrix L in (1.32) is obviously not unitary, although the spatial rotations as in (1.37) are unitary.

We can follow a strategy similar to what is done in nonrelativistic quantum mechanics for the angular momentum algebra

$$[J_i, J_j]=i\epsilon_{ijk}J_k \tag{1.47}$$

In this case, one seeks Casimir operators which commute with all J_i, It turns out that there is only one independent one, namely, J^2. Then we can define eigenstates for the mutually commuting set of operators J^2, J_3. The action of J_1 and J_2 can then be worked out from the commutator algebra (1.47).

For the Poincaré algebra, there are two Casimir operators. One of them is P^2. Since P_μ commute among themselves, we get $[P_\mu, P^2] = 0$. Further

$$[M_{\mu\nu}, P^2] = [M_{\mu\nu}, P_\alpha P_\beta \eta^{\alpha\beta}] = -i\left(\eta_{\mu\alpha}P_\nu P_\beta - \eta_{\nu\alpha}P_\mu P_\beta + \eta_{\mu\beta}P_\alpha P_\nu - \eta_{\nu\beta}P_\alpha P_\mu\right)\eta^{\alpha\beta} = 0 \tag{1.48}$$

Thus P^2 commutes with all operators in the algebra and is thus a Casimir operator. To specify the second one, we first define the operator

Pauli-Lubanski operator

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}P_\nu M_{\alpha\beta} \tag{1.49}$$

This is known as the Pauli-Lubanski operator. Using the Poincaré algebra, it is straightforward to verify that

$$\begin{aligned} [P_\mu, W^\alpha] &= 0 \\ [M_{\mu\nu}, W_\alpha] &= -i\left(\eta_{\mu\alpha}W_\nu - \eta_{\nu\alpha}W_\mu\right) \end{aligned} \tag{1.50}$$

We then see that W^2 is also a Casimir operator.

Consider first defining states with a definite eigenvalue for P^2. Since $P^2 = (P_0)^2 - (P_1)^2 - (P_2)^2 - (P_3)^2$ need not have a definite sign, there are three possibilities:

1. $P^2 = m^2 > 0$
2. $P^2 = -m^2 < 0$
3. $P^2 = 0$

We can make a further refinement here. For the first and third cases, P^2 is positive or zero; then the sign of P_0 is also a Lorentz invariant quantity. This follows from the fact that $L^0_0 = \cosh\Omega$ is always positive and that $\sinh\Omega < \cosh\Omega$. Thus if P_0 is positive or zero to begin with, Lorentz transformations keep P_0 positive. Thus we may further split these possibilities as

1a. $P^2 = m^2 > 0,\ P_0 > 0$
1b. $P^2 = m^2 > 0,\ P_0 < 0$
3a. $P^2 = 0,\ P_0 > 0$
3b. $P^2 = 0,\ P_0 < 0$.

Case 1a will describe particles of mass m, case 3a (actually a subcase of it) will describe particles of zero mass. The other cases are unphysical, corresponding to negative energy (cases 1b, 3b) or faster than light propagation (case 2). We will focus on the physical cases.

The set of mutually commuting operators is given by (P_μ, P^2, W^2). To construct the UIR, we will follow the strategy of Wigner. The idea is to go to a special frame which satisfies the given condition on P^2, define the action of various transformations which preserve that frame and boost back to whatever value of P_μ we need. Consider the case of massive particles first. Since P_μ commute among themselves, we can define the eigenstates $|p\rangle = |p_\mu\rangle$ by

Massive particles

$$P_\mu\ |p\rangle = p_\mu\ |p\rangle \tag{1.51}$$

In this case, the special frame we can choose is the rest frame given by

$$p_\mu \equiv k_\mu = (m, 0, 0, 0) \tag{1.52}$$

This obviously satisfies the conditions $P^2 = m^2$ and $P_0 > 0$. There is a special Lorentz transformation which we will denote by $B^\mu_{\ \nu}(p)$ which will take us from k_μ to an arbitrary value of p_μ obeying the given conditions. This is given explicitly by

$$B^\mu_{\ \nu} = \delta^\mu_{\ \nu} - \frac{(p^\mu + k^\mu)(p_\nu + k_\nu)}{(p \cdot k + k^2)} + \frac{2\,p^\mu k_\nu}{k^2} \tag{1.53}$$

It is trivial to verify that $B^\mu_{\ \nu} k^\nu = p^\mu$. One can also verify that it is a Lorentz transformation in the sense that

$$\eta_{\mu\nu} B^\mu_{\ \alpha}\, B^\nu_{\ \beta} = \eta_{\alpha\beta}, \qquad \text{or} \quad B^\mu_{\ \alpha}\left(\eta_{\mu\nu} B^\nu_{\ \beta} \eta^{\beta\lambda}\right) = \delta^\lambda_\alpha \tag{1.54}$$

(We also see that B is invertible.) Although any Lorentz boost will change the frame, there are still some transformations we can do which preserve k_μ. These are obviously spatial rotations, since the space part of k_μ is zero. Therefore we can define states corresponding to rotations, namely, eigenstates of angular momentum. These are labeled by a j-value (with $J^2 = (M_{12}^2 + (M_{23})^2 + (M_{31})^2 = j(j+1))$ and an eigenvalue for $J_3 = M_{12}$. Being the angular momentum in the rest frame, this is actually the spin of the particle, so we will use s instead of j from now on. In this case,

$$W^2 = \frac{1}{4}(\epsilon^{i0jk} k_0 M_{jk})^2 = m^2 J^2 = m^2 s(s+1) \tag{1.55}$$

(Since W^2 is a Casimir operator, this value will be the same in any frame of reference.) We can label the states as $|s, n\rangle$ where n takes the values $-s$ to s, as usual. The action of rotations is thus given by

$$e^{iJ_i\theta_i}\ |s, n\rangle = \sum_{n'} |s, n'\rangle\ \langle s, n'|\, e^{iJ_i\theta_i}\ |s, n\rangle = \sum_{n'} U_{n'n}(\theta)\ |s, n'\rangle \tag{1.56}$$

U is unitary as in standard angular momentum theory. However, we still need to understand how to represent Lorentz transformations. Wigner's key observation was that one can represent them unitarily by defining a spatial corresponding to every Lorentz transformation.

Wigner rotation

This is the so-called Wigner rotation defined by

$$(R(\theta_W))^\mu_{\ \nu} = (B^{-1}(Lp))^\mu_{\ \alpha}(L)^\alpha_{\ \beta}(B(p))^\beta_{\ \nu} \tag{1.57}$$

Acting on k^ν we find

$$\begin{aligned}(R(\theta_W))^\mu_{\ \nu} k^\nu &= (B^{-1}(Lp))^\mu_{\ \alpha}(L)^\alpha_{\ \beta}(B(p))^\beta_{\ \nu} k^\nu \\ &= (B^{-1}(Lp))^\mu_{\ \alpha}\left[(L)^\alpha_{\ \beta} p^\beta\right] = (B^{-1}(p'))^\mu_{\ \alpha} p'^\alpha, \qquad p' = Lp\end{aligned}$$

$$= \quad k^\mu \tag{1.58}$$

Since k^μ is unchanged, we see that the combination on the right hand side of (1.57) must indeed be a spatial rotation. The angle θ_W can be worked out from (1.57) in terms of the parameters Ω, θ which appear in L. We can represent the Wigner rotation unitarily as in (1.56). The full action is then given, in terms of arbitrary values for p_μ, as

$$U(L)\ |p,s,n\rangle = \sum_{n'} U_{n'n}(\theta_W)\ |Lp,s,n'\rangle \tag{1.59}$$

This gives a unitary action for Lorentz transformations, $U(L)$ being the operator corresponding to L. Translations can obviously act by $e^{iP\cdot a}$ which is unitary since P_μ have real values. To complete the analysis, we need to define an inner product for the states $|p,s,n\rangle$. Notice that we cannot integrate over all values of p_μ using a Euclidean measure d^4p because $p^2 = m^2$ and so we have only three components, say the three spatial ones, which can be chosen freely. To define a Lorentz invariant measure, we use the fact that $p^2 = m^2$ and $p_0 > 0$ are invariant conditions. So consider the combination

$$d\tilde{\mu} = d^4p\, \delta(p^2 - m^2)\, \Theta(p_0) \tag{1.60}$$

where we use a Dirac δ-function and a step function

$$\Theta(p_0) = \begin{cases} 1 & p_0 > 0 \\ 0 & p_0 < 0 \end{cases} \tag{1.61}$$

The combination $d\tilde{\mu}$ is obviously Lorentz invariant. Since

$$\delta(p^2 - m^2) = \frac{1}{2E_p}\big[\delta(p_0 - E_p) + \delta(p_0 + E_p)\big], \qquad E_p = \sqrt{\vec{p}\cdot\vec{p} + m^2} \tag{1.62}$$

we can trivially do the p_0-integration and reduce $d\tilde{\mu}$ to

$$d\mu = d^3p \frac{1}{2E_p} \tag{1.63}$$

This implies that the combination $2E_p\delta^{(3)}(\vec{p}-\vec{p}\,')$ is Lorentz invariant, since its integral with $d\mu$ is 1 for all $\vec{p}\,'$. So we define the inner product for the states $|p,s,n\rangle$ as

$$\langle p,s,n|p',s',n'\rangle = 2E_p\delta^{(3)}(\vec{p}-\vec{p}\,')\,\delta_{ss'}\delta_{nn'} \tag{1.64}$$

To summarize: The states for the massive case are given by $|p,s,n\rangle$ characterized by the mass m, spin s and the 3-momentum $\vec{p}$, with the energy (corresponding to $i\partial/\partial x^0$) given by $E_p = \sqrt{\vec{p}^2 + m^2}$ and having $2s+1$ polarization states or components for spin. This defines a massive point-particle.

The classical view of a point-particle as the idealization of a small extended object of mass m in the limit of zero radius is meaningless in the quantum theory because of position and momentum uncertainties. Also extended bodies do not make sense in relativity because of Lorentz-Fitzgerald contraction effects, so an invariant definition of its extent cannot be given. So the only way to define a point-particle is as a unitary irreducible representation of the Poincaré algebra. Abstract as it may seem this is the only meaningful definition of a point-particle.

Massless particles

Turning to the massless case with $P^2 = 0$, we see that we cannot choose what may be considered a rets frame. However, we can choose a special frame where the spatial momentum is oriented along some particular direction. Thus we may take $k_\mu = E(1, 0, 0, 1)$ which corresponds to energy E and a spatial momentum of magnitude E along the x_3-direction. As with the massive case, we can now ask what transformations will preserve the vector k_μ, so we do not change this frame. These are given by $T_1 = M_{01} + M_{13}$, $T_2 = M_{02} + M_{23}$, and M_{12}. The commutation rules among these are

$$[M_{12}, T_\pm] = \pm T_\pm, \qquad [T_+, T_-] = 0, \qquad T_\pm = T_1 + \pm i T_2 \tag{1.65}$$

We can also check that $W^2 = T_+ T_-$. One class of states is then given by

$$T_+ |p, \lambda\rangle = 0, \qquad T_- |p, \lambda\rangle = 0, \qquad M_{12} |p, \lambda\rangle = \lambda |p, \lambda\rangle \tag{1.66}$$

One can also check that $W^\mu = 2\lambda p^\mu$. This also can be written as

$$\lambda = \frac{W^0}{2p^0} = \frac{\vec{p} \cdot \vec{J}}{p^0} \tag{1.67}$$

λ is referred to as the helicity of the particle. Since it is the eigenvalue of the angular momentum component M_{12}, the possible values of λ are quantized as $0, \pm\frac{1}{2}, \pm 1$, etc. Each value of λ corresponds to one irreducible representation. If we include parity as a symmetry, we will have both $\pm|\lambda|$ since λ changes sign under parity. As in the massive case, one can define a Wigner rotation and write down the unitary realization of the Lorentz transformations. The photon ($\lambda = \pm 1$) and the graviton ($\lambda = \pm 2$) are examples of these representations. Notice that there are only two spin states or two polarization states (if parity is a symmetry) for massless particles for any value of helicity. The left ($\lambda = 1$) and right ($\lambda = -1$) circular polarizations of the photon are precisely these two helicity states.

One can also construct representations where the eigenvalues for $T_\pm$ are not zero. So far, they do not seem to have any physical significance.

1.4 Why do we need quantum field theory?

We have obtained the relativistic and quantum theoretic description of a point-particle. We can, as a trivial generalization, consider a system of N noninteracting particles. (Choose N

copies of the appropriate (massive or massless) representations of the Poincaré algebra.) So the next question we can ask is whether we can build in interactions. In Newtonian mechanics, this is straightforward. We are familiar with interactions in terms of potentials $V(x^{(\alpha)}, x^{(\beta)})$ between particles identified by the labels as α and β, at positions $x^{(\alpha)}$ and $x^{(\beta)}$. But once we include the principle of relativity, this runs into trouble. It is hard to imagine a potential which allows for relativistic invariance but is independent of time or time-differences. But if have a time-dependent potential, the Hamiltonian description becomes problematic. This can be made more precise by the so-called no-interaction theorems. The best known such theorem is due to Currie, Jordan and Sudarshan. There are three assumptions which form the premise of this theorem. They are:

No-interaction theorem

1. The world lines of particles, say given as $x^\mu(\tau)$ for some parameter τ, is the same viewed from different frames of reference modulo Poincaré transformations; i.e., for the infinitesimal case,

$$(x^\mu)'(\tau) \approx x^\mu(\tau + \delta\tau) + \omega^{\mu\nu} x_\nu(\tau + \delta\tau) + a^\mu \tag{1.68}$$

   (A small shift in τ is also generally allowed since this does not change the world line itself.)
2. Poincaré transformations can be realized as canonical transformations (anticipating that they will be represented as unitary transformations upon quantization).
3. Positions of particles are canonical variables in the sense that they are mutually Poisson-commuting; i.e, the Poisson brackets are

$$\{x_i^{(\alpha)}, x_j^{(\beta)}\} = 0 \qquad \text{for all } \alpha, \beta = 1, 2, \cdots, N. \tag{1.69}$$

   (This is also in expectation that the particle positions commute as operators in the quantum theory.)

All these assumptions are eminently reasonable; we need a canonical formalism to carry out the usual procedure of quantization. The theorem is then:

> **Theorem 1.2 — Currie-Jordan-Sudarshan theorem.** Given the three conditions stated above, the only mechanical system of N particles is the system of N free particles with no interaction among them.

Although one might try to evade the theorem by giving up on one or more of the conditions listed as the premise (and people have tried doing that), if we want to have these reasonable properties, the only solution is to have interactions mediated by fields, which themselves can propagate with a finite speed $\leqslant c$. By going beyond particle mechanics by the inclusion of fields, the theorem is easily evaded. And this is indeed the case in the classical interacting theories we know, where the forces are mediated by the electromagnetic field or by the gravitational field (within the framework of the general theory of relativity).

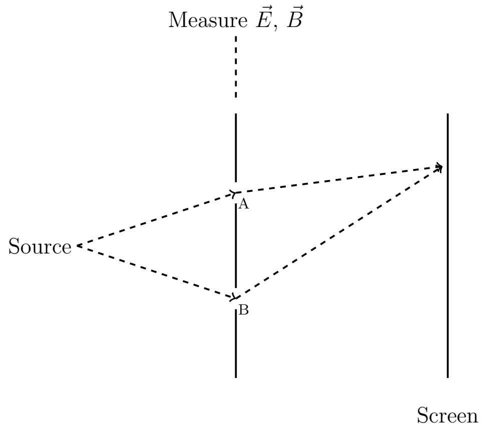

Figure 1.1: The double slit experiment with electrons

Once we accept that we must have fields based on the no-interaction theorem, or simply accept the fact that the electromagnetic field exists, we can ask the question about the quantum theory. One possibility one might entertain is that one needs to quantize the mechanics of particles, to get the good results for the electrons in atoms and so on, but keep the fields classical. This is inconsistent. It is possible to construct *gedanken* experiments which show that one can then avoid the uncertainty principle and thus get inconsistency within particle quantum mechanics. Bohr and Rosenfeld have analyzed the measurement problem in some detail to argue for quantizing the fields. A simple argument (also due to Bohr) is the following. Consider the double slit interference experiment carried out using electrons as shown schematically in Fig. 1.1. By virtue of the matter wave idea we should expect, and indeed we do find experimentally as well, an interference pattern. But the particles also have electric and magnetic fields associated to them. Consider measuring these fields at some distance away from the set-up. Also assume that we can measure both the electric and magnetic fields with no uncertainty between them, as would be the case if they are classical fields. The measurement is carried out closer to slit A than slit B, so the fields would be slightly stronger if the electron went through A. Therefore from a precise measurement, we can determine which slit the electron went through. In this case there should be no interference pattern, since there no amplitude for the electron to go through the other slit, in contradiction with what we expect from the quantum mechanics of particles. However, if we have an uncertainty relation for the fields themselves, namely, we cannot determine both $\vec{E}$ and $\vec{B}$ precisely, we lose the possibility of determining which slit the electron went through. This tells us that consistency of quantum mechanics even at the level of particles will require the quantization of fields which are coupled to them.

Need to quantize fields

There is another way to look at quantum field theory. We can describe the quantum mechanics of N particles using the usual many-body wave functions, a Hamiltonian for the particles, etc. However there exist processes where the particle number is not conserved, for example, the β-decay of a neutron into a proton, electron and an antineutrino, $n \to p + e^- + \bar{\nu}_e$. The neutron wave function ψ_n must obey $\int d^3x\, \psi_n^* \psi_n = 1$ before decay since the neutron must be somewhere, and $\int d^3x\, \psi_n^* \psi_n = 0$ after decay. Likewise $\int d^3x\, \psi_p^* \psi_p$, $\int d^3x\, \psi_e^* \psi_e$, etc. are not conserved. The usual formalism of quantum mechanics cannot accommodate this, we have to augment the formalism of many-body quantum mechanics to include the possibility of particle decay and particle creation. In the relativistic case where the interconversion of mass and energy is possible, this is unavoidable. This augmented formalism is quantum field theory.

QFT= Many-body QM + creation & annihilation

1.5 Fields, wave functions, gauge symmetry

We have given arguments as to why we need quantum field theory. A natural question then might be: If we have to introduce fields and quantize them, then why did we bother with representations of the Poincaré algebra? The answer to this question has two important parts.

First of all, in quantum field theory, single particle wave functions do correspond to the representations we have discussed. As an example, consider the relativistic scalar free scalar field described by the Klein-Gordon equation

$$\left(\Box + m^2\right) \phi = 0, \qquad \Box = \frac{\partial^2}{\partial t^2} - \nabla^2 \tag{1.70}$$

Let f_1 and f_2 be two solutions; these are of the form

One-particle wave functions

$$f = C\, e^{-ip\cdot x} = C\, e^{-ip_0 x^0 + i\vec{p}\cdot\vec{x}}, \qquad (p_0)^2 - \vec{p}\cdot\vec{p} - m^2 = 0 \tag{1.71}$$

The fact that ϕ is a quantum field, i.e., an operator means that the coefficient C is an operator. We then find

$$\eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \left(f_1^* \frac{\partial}{\partial x^\nu} f_2 - \frac{\partial f_1^*}{\partial x^\nu}, f_2 \right) = 0 \tag{1.72}$$

Integrating this over the spatial directions and taking the boundary condition [1]

$$\oint dS\, \left(f_1^* n\cdot\nabla f_2 - n\cdot\nabla f_1^*\, f_2\right) = 0, \tag{1.73}$$

we find that

$$\frac{\partial}{\partial t} \int d^3x\, \left(f_1^* (\partial_0 f_2) - (\partial_0 f_1^*) f_2\right) = 0 \tag{1.74}$$

[1] This is a sensible boundary condition; it can be satisfied either by the vanishing of the f's at spatial infinity or by periodic boundary conditions.

Thus the inner product which is preserved in time is

$$\langle f_1|f_2\rangle = i\int d^3x\, (f_1^*\partial_0 f_2 - (\partial_0 f_1^*)f_2) \tag{1.75}$$

Using (1.71), we see that this amounts to

$$f(x) = C\,\langle x|p\rangle\,, \qquad \langle p'|p\rangle = 2E_p\,\delta^{(3)}(\vec{p}-\vec{p}\,') \tag{1.76}$$

in agreement with our earlier discussion. Thus for free fields, and for interacting fields within perturbation theory, we do get the UIRs of the Poincaré algebra.

Need for gauge symmetry

Secondly, our analysis provides a window into gauge theories. Consider a theory of a massless particle like the photon. This has helicity equal to ± 1. Since this is the eigenvalue of M_{12}, the spin values along the direction of the momentum is ± 1. A 3-vector A_i has spin 1, so this is the kind of field we need to write a field theory for this. But relativistic invariance requires all four components A_μ. This leads to a mismatch since the analysis of the Poincaré algebra for massless particles shows that we have only two polarizations. How do we deal with this? We must make sure that there is some additional symmetry which makes two components (out of the 4 in A_μ) redundant and unphysical, leaving us with two polarizations. This required symmetry related to redundancy is the gauge symmetry for the electromagnetic field. Notice that this has to be a local symmetry, since we need to remove two x-dependent fields of the four $A_\mu(x)$.

The situation is even more drastic for the graviton which has helicity ± 2. This means that we need a symmetric tensor h_{ij} to provide the required spin-2 fields. Promoting this to $h_{\mu\nu}$ in four dimensions, we get 10 components. So a gauge symmetry is needed to reduce this to the two physical polarizations. This symmetry is the freedom of local coordinate transformations in the general theory of relativity.

1.6 Conformal symmetry

We now turn to what are called conformal transformations. These are transformations which do not necessarily preserve the metric, but they leave the metric unchanged up to an overall multiplicative factor. For an infinitesimal change $x^\mu \to x^\mu + \xi^\mu$, this means that, instead of (1.16), we should have

$$(ds^2)_{x+\xi} \approx (1+\lambda)(ds^2)_x \tag{1.77}$$

where λ is also infinitesimal, of the order of ξ. Using (1.17), we then find

$$\xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\alpha\nu}\frac{\partial \xi^\alpha}{\partial x^\mu} + g_{\mu\alpha}\frac{\partial \xi^\alpha}{\partial x^\nu} = \lambda\, g_{\mu\nu} \tag{1.78}$$

which we may write in the more covariant notation as

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \lambda\, g_{\mu\nu} \tag{1.79}$$

Let us consider this n dimensions to begin with. Then contracting with $g^{\mu\nu}$ and using $g^{\mu\nu}g_{\mu\nu} = n$ we get

$$\lambda = \frac{2}{n}\nabla_\alpha \xi^\alpha \tag{1.80}$$

This eliminates λ and we can write (1.79) as

Conformal Killing equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{2}{n}(\nabla_\alpha \xi^\alpha) g_{\mu\nu} = 0 \tag{1.81}$$

This is known as the conformal Killing equation. Since the isometries do not change the metric, they are automatically solutions of the conformal Killing equation as well. But there are additional solutions possible.

Let us consider flat Minkowski space again. In this case the equation becomes

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{1}{2}(\partial \cdot \xi)\,\eta_{\mu\nu} = 0 \tag{1.82}$$

A scale transformation of all coordinates as $x^\mu \to (1+\epsilon)x^\mu$, or $\xi^\mu = \epsilon x^\mu$ is a solution since $\partial_\mu \xi_\nu = \epsilon \eta_{\mu\nu}$ and $\partial \cdot \xi = 4$. This is referred to as a dilatation. Another set of solutions is given by the so-called special conformal transformations which are quadratic in the x's; they are given by

$$\xi^\mu = b^\mu x^2 - 2b_\alpha x^\alpha x^\mu \tag{1.83}$$

It is easy to verify that this is indeed a solution. By considering a power series as we did for the isometries, one can show that there are no other solutions. The special conformal transformations can also be viewed as translations of the "inverted coordinate"

$$y^\mu \to y^\mu + b^\mu, \qquad x^\mu = \frac{y^\mu}{y^2} \tag{1.84}$$

So the conformal isometries of the Minkowski metric are given by

Conformal isometries of Minkowski space

$$\xi^\mu = a^\mu + \omega^{\mu\nu}x_\nu + \epsilon\, x^\mu + b^\mu x^2 - 2b_\alpha x^\alpha x^\mu \tag{1.85}$$

We can thus define 15 generators for the conformal transformations in 4 dimensions; their algebra is the conformal algebra in four dimensions.

If we consider n dimensions, then we have the same pattern with n translations (a^μ), n inverted translations or special conformal transformations (b^μ), $\frac{1}{2}n(n-1)$ Lorentz transformations (for the antisymmetric $\omega^{\mu\nu}$) and one dilatation ϵ, giving $\frac{1}{2}(n+1)(n+2)$ parameters. However, there is one exception to this which is for $n = 2$. Consider two dimensional space $\mathbb{R}^2$ with Euclidean signature. Then the components of the conformal Killing equation are

$$\partial_1 \xi_1 - \partial_2 \xi_2 = 0, \qquad \partial_1 \xi_2 + \partial_2 \xi_1 = 0 \tag{1.86}$$

In terms of complex coordinates

$$x^1 + ix^2 \equiv z, \quad x^1 - ix^2 = \bar{z}, \quad \frac{\partial}{\partial z} = \tfrac{1}{2}(\partial_1 - i\partial_2), \quad \frac{\partial}{\partial \bar{z}} = \tfrac{1}{2}(\partial_1 + i\partial_2) \tag{1.87}$$

equations (1.86) become

$$\frac{\partial}{\partial \bar{z}}\xi = 0, \quad \frac{\partial}{\partial \bar{z}}\bar{\xi} = 0, \quad \xi = \xi^1 + i\xi^2, \quad \bar{\xi} = \xi^1 - i\xi^2 \tag{1.88}$$

This means that any holomorphic transformation $z \to z + \xi(z)$ is a conformal transformation. If we remove the points at $z = 0$ and $z \to \infty$, then we can do a Laurent expansion of ξ as

Conformal isometries in 2 dim.

$$\xi(z) = -\sum_{n=-\infty}^{\infty} z^{n+1} a_n \tag{1.89}$$

for arbitrary parameters a_n. We see that the two-dimensional conformal isometries will lead to an infinite dimensional algebra. Since the transformation of any holomorphic function takes the form $f(z) \to f(z) + \xi\partial_z f(z)$, we can write the generators as

$$L_n = -z^{n+1}\frac{\partial}{\partial z} \tag{1.90}$$

The commutator bracket becomes

$$[L_m, L_n] = (m - n)\, L_{m+n} \tag{1.91}$$

This is known as the Witt algebra. When we construct quantum field theories with this symmetry, the algebra usually gets modified by quantum corrections and takes the general form

Virasoro algebra

$$[L_m, L_n] = (m - n)\, L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{m+n,0} \tag{1.92}$$

This is known as the Virasoro algebra and the extra term is a "central extension". The parameter c is known as the central charge.

Let us now take up the question of why these symmetries can be important.

CFT & Phase transitions

1. Classically, one can construct field theories which have conformal symmetry. The electromagnetic field in vacuum (free space Maxwell equations) and the electromagnetic field coupled to massless charged fields are examples. However, generally, conformal symmetry will not survive quantization. But there can be special values of the coupling constants and other parameters where a quantum field theory can have conformal symmetry. Physically, these correspond to phase transitions which are of the second order (or higher). Field theories at such critical points, which we can refer to as conformal field theories, are thus important to the study of phase transitions. The Ising model of magnetism at its critical point is an example.

2. There are also special theories which have conformal symmetry for all values of the couplings, or for a significant range of couplings. The best known example is the maximally supersymmetric Yang-Mills theory in four dimensions. This is a key component of the so-called AdS/CFT conjecture.
   The 15-parameter conformal symmetry of Minkowski space can be realized as the isometry (not conformal isometry, but actual isometry) of the anti-de Sitter (AdS) space in five dimensions. The conjecture (not proven, but is supported by a large number of calculations) then says that the maximally supersymmetric four-dimensional Yang-Mills theory can be used to define string theory on the 10-dimensional space $AdS^5 \times S^5$. Since string theory does have gravity in it, the hope is that this correspondence can be used to understand quantum gravity.

AdS/CFT conjecture

3. Two-dimensional conformal field theories are special. In higher dimensions, while the conformal algebra is realized as a subalgebra of the full algebra of observables $\mathcal{A}$ at critical points, the theory has a large number of observables and the full dynamics cannot be obtained by just using conformal symmetry, which has only $\frac{1}{2}(n+1)(n+2)$ parameters. But in two dimensions we have an infinite number of parameters and one may hope to completely specify a theory by representations of the Virasoro algebra. Indeed this is the case for certain values of c given by

2d phase transitions

$$c = 1 - \frac{6}{m(m+1)}, \qquad m = 2, 3, \cdots \tag{1.93}$$

   The theory can be completely worked out just from the algebra (1.92). The corresponding critical points in two dimensions can thus be completely understood. The 2d Ising model and a number of other models from statistical mechanics belong to this set. Even other values of c, say $c > 1$ can be understood by considering a larger algebra such as a Kac-Moody algebra; there is a well-understood construction of the Virasoro algebra from the Kac-Moody algebra.
4. There is yet another reason why two-dimensional conformal field theories are important. Since a string has a spatial extent as well as progression in time, a string traces out a two-dimensional surface in spacetime, a worldsheet rather than a worldline. The quantum dynamics of a single string can be viewed as as a two-dimensional conformal field theory on the worldsheet. A lot of information regarding scattering of strings can be obtained by analyzing the propagation of a single string, even though the full theory must allow for multiple strings as well as processes of creating and annihilating strings, i.e., one needs to use a string field theory.

String as a 2d CFT

1.7 Appendix: The stereographic projection

The stereographic projection of the two-sphere is an interesting result in its own right and will be useful to us later, but it also illustrates how one can have very different coordinate systems describing the same manifold and how the isometries are independent of the coordinates used.

The standard and well-known set of coordinates is (θ, φ) corresponding to $\theta = \frac{\pi}{2} -$ latitude and φ as the azimuthal (longitude) angle. These are related to the Cartesian coordinates (X_1, X_2, X_3)

$$X_1 = r \sin\theta \, \cos\varphi, \quad X_2 = r \sin\theta \, \sin\varphi, \quad X_3 = r\cos\theta \tag{1.94}$$

which describe the embedding of the sphere S^2 in $\mathbb{R}^3$.

The stereographic projection is obtained as follows. We consider the equatorial plane of a two-sphere. A line from the north pole of the sphere is extended through the point of interest, denoted $\hat{n}$ in Fig. 1.2. This line meets the point (ρ, φ) on the plane. We can use complex coordinates for the plane given by z, $\bar{z}$, with $z = \rho e^{i\varphi}$. $(z, \bar{z})$ can be used as coordinates for the point $\hat{n}$. From the figure shown on the right in Fig. 1.2, we see that $\rho = r\tan\alpha$, with $\alpha = \frac{1}{2}(\pi - \theta)$. This gives

$$\rho = r\cot(\theta/2) \tag{1.95}$$

From (1.94), we see that

$$X_3 = r\,\frac{\rho^2 - r^2}{\rho^2 + r^2} = r\,\frac{(\bar{z}z/r^2) - 1}{(\bar{z}z/r^2) + 1} \tag{1.96}$$

Also, working out $\sin\theta$, we find

$$X_1 = \frac{(z + \bar{z})}{(\bar{z}z/r^2) + 1}, \quad X_2 = -\frac{i(z - \bar{z})}{(\bar{z}z/r^2) + 1} \tag{1.97}$$

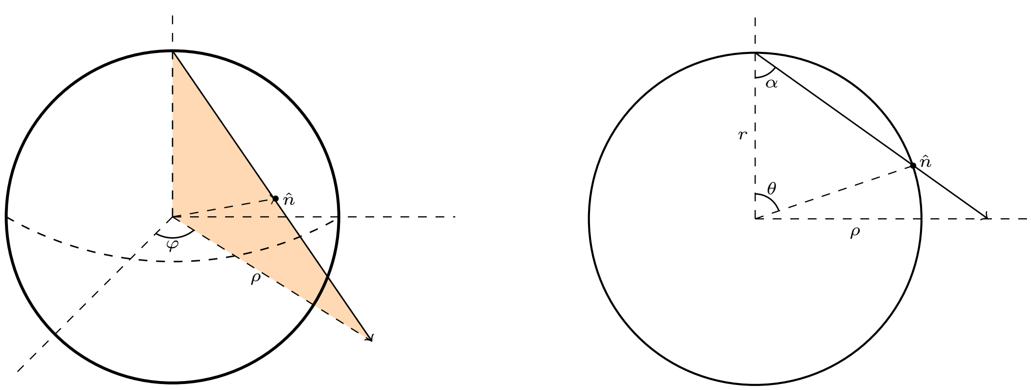

Figure 1.2: Stereographic projection of a sphere on to the plane. The point labeled $\hat{n}$ on the sphere is projected to the point on the plane with polar coordinates (ρ, φ). The dashed lines and the coloring of the projection plane are to guide the eye. On the right side is the transverse view of the plane containing the projection line.

For simplicity, we will now set $r = 1$; it can be recovered by rescaling $z, \bar{z}$. The above equations then become

$$X_1 = \frac{(z+\bar{z})}{\bar{z}z+1}, \quad X_2 = -\frac{i(z-\bar{z})}{\bar{z}z+1}, \quad X_3 = \frac{\bar{z}z-1}{\bar{z}z+1} \tag{1.98}$$

In this stereographic projection, the north pole of S^2 corresponds to $\bar{z}z \to \infty$, which is the point $(X_1, X_2, X_3) = (0, 0, 1)$. Notice that the whole circle at $\bar{z}z \to \infty$ of the plane is mapped to one point, namely $(0, 0, 1)$ on the sphere.

The metric of the sphere is given in the (θ, φ) as

$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 \tag{1.99}$$

In terms of the complex coordinates z, $\bar{z}$, this is given by

$$ds^2 = 4\,\frac{dz d\bar{z}}{(1+\bar{z}z)^2} \tag{1.100}$$

There are three isometries for the two-sphere. These are given in the (θ, φ)-system as

$$\begin{aligned} \theta &\to \theta + a_1 \sin\varphi - a_2 \cos\varphi \\ \varphi &\to \varphi + a_1 \cot\theta \cos\varphi + a_2 \cot\theta \sin\varphi - a_3 \end{aligned} \tag{1.101}$$

One can verify by direct substitution that the metric in unchanged under these changes. The generators of these changes may be written as

$$\begin{aligned} L_1 &= -\sin\varphi \frac{\partial}{\partial\theta} - \cot\theta \, \cos\varphi \frac{\partial}{\partial\varphi} \\ L_2 &= \cos\varphi \frac{\partial}{\partial\theta} - \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \\ L_3 &= \frac{\partial}{\partial\varphi} \end{aligned} \tag{1.102}$$

It is easy to verify that these give the angular momentum algebra (up to a factor of $-i$), showing that the isometries correspond to rotations of the 3d coordinates if we think of the sphere as embedded in $\mathbb{R}^3$. There is no surprise here, since we know the sphere looks the same if we rotate it around any axis; that is its defining property.

One can also check that the isometries are given in terms of the complex coordinates as

$$\begin{aligned} L_1 &= \frac{i}{2}\left[(z^2-1)\frac{\partial}{\partial z} - (\bar{z}^2-1)\frac{\partial}{\partial \bar{z}}\right] \\ L_2 &= \frac{1}{2}\left[(z^2+1)\frac{\partial}{\partial z} + (\bar{z}^2+1)\frac{\partial}{\partial \bar{z}}\right] \\ L_3 &= i\left[z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right] \end{aligned} \tag{1.103}$$

One can verify that these are exactly the same as what is given in (1.102) is we make the change of variables $z = \cot(\theta/2)\, e^{i\varphi}$. Thus the isometries are the same in the two coordinates systems, although their expressions but are simply re-expressed in terms of the new coordinates when we do a coordinate transformation.

1.8 References

1. Isometries and the Killing equation are discussed in most books on the general theory of relativity. Versions which do not require too much mathematical formalism are in: S. Weinberg, *Gravitation and Cosmology*, John Wiley and Sons (1972);
   M. Carmeli, *Classical Fields: General Relativity and Gauge Theory*, John Wiley and Sons (1982).
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3. The no-interaction theorem referred to in text is due to D.G. Currie, T.F. Jordan and E.C.G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963).
4. A nice review of various no-interaction theorems is S. Chelkowski, J. Nietendel and R. Suchanek, Acta Physica Polonica, **B11**, 809 (1980).
5. A review of Bohr's double-slit argument, with updated considerations, is G. Baym and T. Ozawa, Proc. Nat. Acad. Sci. **106**, 3035 (2009).