PHY 552: Practice Problems

A number of problems are given below. The answers to these are not to be submitted. However, I urge you to do these as practice problems, they can help significantly in understanding quantum mechanics better.

1. Consider the one-dimensional 'half-oscillator' potential

$$V(x) = \begin{cases} \infty & x < 0 \\ \frac{1}{2}m\omega^2 x^2 & x > 0 \end{cases} \tag{1}$$

Obtain the bound state energy eigenvalues in the WKB approximation, using the Bohr-Sommerfeld quantization rule.

2. For two arbitrary vectors $|u\rangle$, $|v\rangle$ of a Hilbert space, obtain the Schwarz inequality

$$|\langle u|v\rangle|^2 \leq \langle u|u\rangle \langle v|v\rangle \tag{2}$$

Also obtain the triangle inequality

$$\sqrt{\langle \alpha|\alpha\rangle} \leq \sqrt{\langle u|u\rangle}\sqrt{\langle v|v\rangle} \tag{3}$$

where $|\alpha\rangle = |u\rangle + |v\rangle$.

3. If A, B are two linear operators on a Hilbert space, show that

$$e^A \, B \, e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots + \frac{1}{n!}[A, [A, [A, ...[A, B]...]]] + \cdots \tag{4}$$

where the n-th term involves n successive commutators with A's.

4. If U is a unitary matrix, show that the absolute value of every matrix element is less than or equal to 1, i.e., $|U_{ij}| \leq 1$ for any i, j.

5. The Hamiltonian for a three-dimensional system is of the form

$$H = \frac{p_i p_i}{2m} + V(r) \tag{5}$$

where V depends only on r. Obtain the commutator $[x_k p_k, H]$.

6. Obtain the transmission and reflexion coefficients for particles incident from the right on the potential

$$V(x) = \begin{cases} V_1 & x < 0 \\ 0 & 0 < x < a \\ V_2 & x > a \end{cases} \tag{6}$$

where $V_1 > V_2$ and the particles have energy E with $V_2 < E < V_1$.

7. We define the operators

$$
\begin{aligned}
F_1 &= \frac{1}{4}\left(p^2 + x^2\right) \\
F_2 &= \frac{1}{4}\left(p^2 - x^2\right) \\
F_3 &= \frac{1}{4}\left(xp + px\right)
\end{aligned} \tag{7}
$$

Obtain the commutation relations among these operators and show that they form a closed algebra.

8. The Hamiltonian for a three-dimensional particle is of the form

$$H = \frac{p_i p_i}{2m} + V(\vec{x}) \tag{8}$$

If the particle is in a state $|\alpha\rangle$, calculate the rate of change (time-derivative) of the expectation value of the angular momentum operator L_i and compare with the classical Newtonian law.

9. From the matrix form of the Hamiltonian for the harmonic oscillator calculate $Tr\ \left(e^{-\beta H}\right)$, where β is a constant and Tr denotes the trace or the sum over the diagonal elements. (This quantity is the thermodynamic partition function for a single oscillator.)

10. A particle, in a harmonic oscillator well, is prepared, *at time t=0*, in the state $|\alpha\rangle$ with $\langle x|\alpha\rangle = \frac{1}{2}C\, e^{-\frac{1}{2}\xi^2}(1 + \sqrt{6}\,\xi)$, $\xi = x\sqrt{m\omega/\hbar}$.
a) Determine the normalization constant C.
b) What is the probability of finding the particle at position x at time $t = 0$?
c) What is the probablity of finding the particle at x at time $t = T$?
d) What is the probability that a momentum measurement at time $t = 0$ will give the value k?
e) What is the probability that a momentum measurement at time $t = T$ will give the value k?

11. The quantum mechanics of a rigid rotating body (such as certain molecules, or atomic nuclei) can be described by the Hamiltonian

$$H = \frac{1}{2I}\, L_i L_i \tag{9}$$

where I is the moment of inertia (a constant for our purposes) and L_i are the orbital angular momentum components. Obtain the eigenvalues and eigenstates of the Hamiltonian.

12. The Hamiltonian for the idealized Hydrogen atom is

$$H = \frac{p_i p_i}{2m} - \frac{Ze^2}{r} \tag{10}$$

Show that the Runge-Lenz vector

$$R_i = \frac{1}{2m}\left(\epsilon_{ijk} p_j L_k - \epsilon_{ijk} L_j p_k\right) - e^2 \frac{x_i}{r} \tag{11}$$

commutes with H.

13. Consider the three orbital angular momentum states $|1,1\rangle,\ |1,0\rangle,\ |1,-1\rangle$ in the $|j,m\rangle$-notation. In each case, obtain the probability that the particle is located in the cone $0 \leq \theta \leq \pi/4$. Calculate also the expectation value of L_1^2 for each of these states.

14. The energy eigenstates of the Hydrogen atom are specified by the principal quantum number n, the angular momentum value l and the azimuthal quantum number (L_3-eigenvalue) m. We may write the states as $|n,l,m\rangle$. A particle is prepared in the state

$$|\alpha\rangle = a|2,0,0\rangle + b_1|2,1,1\rangle + b_2|2,1,0\rangle \tag{12}$$

where a, b_1, b_2 are constants. How are these constants constrained? What is the probability of finding the value 1 for a measurement of L_3? What is the probability of finding the value -2 for a measurement of L_3?

15. The ground state wavefunction for the Hydrogen atom is given by

$$\langle\vec{x}|100\rangle = \psi_{100}(r) = \left(\frac{Z^3}{\pi a^3}\right)^{\frac{1}{2}} e^{-Zr/a} \tag{13}$$

where $a = \hbar^2/me^2$. Calculate the uncertainty in the measurement of the radial distance r, viz., Δr^2, for an electron in this state. Calculate also the uncertainty in the measurement of the Cartesian coordinate $z = x_3$ for this state.

16. The inner product or scalar product of spherically symmetric wave functions (depending only on r) is given by

$$\langle 1|2\rangle = \int dr\ r^2\ \psi_1^*\psi_2 \tag{14}$$

Determine which, if any, of the following operators, given as differential operators in spherical polar coordinates, is self-adjoint.

$$A = -i\hbar\frac{\partial}{\partial r}, \qquad\qquad B = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) \tag{15}$$

Which of these would be a suitable candidate for the radial component of the momentum?

17. A particle is constrained to move on a circle of radius R. If the angle along the circle is taken as φ, the Hamiltonian is given by

$$H = -\frac{1}{2mR^2}\frac{\partial^2}{\partial\varphi^2} \tag{16}$$

Determine the energy eigenvalues and eigenstates.

18. A one-dimensional free particle is given, at time $t = 0$, to be in the state $|\alpha\rangle$ described by the momentum space wavefunction

$$\langle p|\alpha\rangle = Cp^2 \exp\left(-\frac{p^2}{4\sigma^2}\right) \tag{17}$$

a) Determine C from the normalization condition.

b) Obtain the $coordinate\ space$ wavefunction for the particle at time $t = T$.

19. We consider an electron in a spherically symmetric Gaussian potential well; the Hamiltonian is given by

$$H = \frac{p_i p_i}{2m} - V_0 e^{-\lambda r^2} \tag{18}$$

where $V_0 > 0$ and λ is a fixed positive number. The electron is given to be in a state with wave function

$$\psi = \left(\frac{2\alpha}{\pi}\right)^{\frac{3}{4}} e^{-\alpha r^2} \tag{19}$$

where α is a positive number. Calculate the expectation value of the energy in this state. The expectation value of the energy is minimum for a particular value of α, equal to α^*, say. Obtain the relation between α^* and λ. (You don't have to solve this relation to obtain α^*.)

20. A particle in a harmonic oscillator potential is prepared to be in the quantum state given by

$$\psi(x) = \left[\frac{16m\omega}{9\pi\hbar}\right]^{\frac{1}{4}} \xi^2 \exp(-\tfrac{1}{2}\xi^2) \tag{20}$$

where $\xi = x\sqrt{m\omega/\hbar}$.

a) If a measurement is now carried out on this system, what are the probabilities of finding the particle in the ground state ψ_0, the first excited state ψ_1 and second excited state ψ_2? Recall that the energy eigenstates of the oscillator are given by

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}\ H_n(\xi) \tag{21}$$

b) What is the expectation value of the energy for this state?

21. A particle is given to be in the ground state of a harmonic oscillator with wave function

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} \tag{22}$$

where $\xi = x\sqrt{m\omega/\hbar}$. What is the probability of obtaining a value of momentum equal to k if a momentum measurement is carried out?

22. The potential near the surface of a metal acted on by an external electric field E can be approximated as follows.

$$V(x) = \begin{cases} -V_0 & x < 0 \quad \text{(inside)} \\ C - eEx & x \geq 0 \quad \text{(outside)} \end{cases} \tag{23}$$

$V_0,\ C$ are constants, e is the electronic charge. Calculate, in the WKB approximation, the probability that an electron in the metal (taken to be a plane wave) can tunnel out under the action of the electric field.

23. We want to determine the lowest energy eigenstate of a particle in a one-dimensional potential well

$$V(x) = -\lambda\,(sech\ x)^2 \tag{24}$$

This can be done as follows. Using the fact that the derivative of $tanh\ x$ is $(sech\ x)^2$, show that the Hamiltonian can be written as

$$H = \frac{p^2}{2m} + V(x) = \rho^\dagger \rho - \alpha^2 \tag{25}$$

where $\rho = (\frac{1}{2m})^{\frac{1}{2}}(p + if),\ f(x)$ being a suitably chosen function and α is a constant. Determine $f,\ \alpha$. Now any operator of the form $\rho^\dagger\rho$ is positive semi-definite. Using this fact, determine the lowest energy eigenvalue and the corresponding eigenfunction for this system.

24. Obtain the transcendental equation which gives the bound state eigenvalues of a particle in the one-dimensional potential well

$$V(x) = \begin{cases} \infty & x < 0 \\ -V_0 & 0 < x < a \\ 0 & x > a \end{cases} \tag{26}$$

Obtain also the normalized wave functions.

25. Since the components of angular momentum do not commute among themselves, their simultaneous measurement is not possible. Calculate the uncertainty in J_2 for the states $|j, m\rangle$ and find the minimu value of m (for a fixed j) which minimizes this uncertainty.

26. Consider a charged spin-$\frac{1}{2}$ particle like the electron which is placed in a magnetic field. Obtain the Heisenberg equations of motion for its spin vector.

27. An electron beam is prepared in the $S_3 = \frac{1}{2}$ state at time $t = 0$. The beam then enters a uniform magnetic field $B_i = B(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. What is the probability of obtaining $S_1 = \frac{1}{2}$ at time t?

28. A beam of spin-1 particles are prepared in a state $S_3 = +1$. A measurement of the operator $K = \frac{1}{\sqrt{2}}\,(S_1 + S_2)$ is then carried out. What is the probability of obtaining $K = -1$?

29. Calculate the differential and total cross sections for the following potentials in the Born approximaton.
   1. $V(r) = \frac{A}{r}e^{-ar}$
   2. $V(r) = V_0 e^{-a^2r^2}$

30. Find the phase shifts and the differential cross section for the scattering of slow particles by the potential $V(r) = A/r^2$. What happens when $2mA\hbar^{-2} \ll 1$.

31. Consider two spin-1 particles in orbital states described by $f(x)$ and $g(x)$. Find the possible two-particle states of the system.

32. Find the energy spectrum of a system whose Hamiltonian is given by

$$H = H_0 + H' = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2 x^2}{2} + ax^3 + bx^4 \tag{27}$$

   where a, b are 'small' constants.

33. Consider a double pendulum, with segments of lengths L_1 and L_2 and bobs of masses m_1 and m_2, in the small oscillations approximation familiar from classical mechanics. Obtain the quantized energy levels of the system.

34. In the preceding problem, if there is an additional potential of the form $V = \lambda(z_1 + z_2)$ where z_i are the heights of then two bobs measured from the point of suspension, find the correction to the energy levels to first order in λ.

35. The quantum dynamics of a ball of mass m bouncing perfectly elastically on the ground may be described by the Hamiltonian

$$H = \frac{p^2}{2m} + mgx \tag{28}$$

   where x is the height. Use a variational ansatz of the form

$$\Psi = Cxe^{-ax} \tag{29}$$

   to estimate the ground state energy. What is not right about using an ansatz of the form $\Psi = Ce^{-ax}$ without the extra factor of x?

36. The Morse potential, frequently used in discussions of the vibrational spectra of molecules, is given by

$$V(x) = V_0\left(1 - e^{-\mu x}\right)^2 \tag{30}$$

   where $-\infty \le x \le \infty$. Using a harmonic oscillator type ansatz of the form $\Psi = C\exp(-\alpha x^2)$, obtain the variational estimate of the ground state energy.

37. A charged particle harmonic oscillator is placed in a time-dependent homogeneous electric field given by

$$\mathcal{E}(t) = \frac{A}{\sqrt{\pi}\,\tau}\, e^{-\frac{t^2}{\tau^2}} \tag{31}$$

where A, τ are constants. If the oscillator is in its ground state at time $t = -\infty$, find the probability, to the first approximation, that it will be in the first excited state at time $t = \infty$.

38. The β^+ decay of an atomic nucleus of charge Z causes a sudden change in the nuclear charge $Z \to Z - 1$, whose effect on the electron may be represented as a perturbation

$$V = \begin{cases} 0 & t < 0 \\ e^2/r & t > 0 \end{cases} \tag{32}$$

To first order in this perturbation, calculate the probability that the electron will make a transition from the ground state to the $2S$ state.

39. Consider a single harmonic oscillator of frequency ω. Calculate the second order correction to the ground state energy due to the perturbation $V = \lambda x^4$. (In class, I evaluated the first order correction.)

40. We consider the isotropic two-dimensional harmonic oscillator with

$$\mathcal{H} = \frac{p_1^2}{2m} + \frac{m\omega^2 x_1^2}{2} + \frac{p_2^2}{2m} + \frac{m\omega^2 x_2^2}{2} \tag{33}$$

We can construct eigenstates using

$$\begin{aligned} a_1 &= \sqrt{\frac{m\omega}{2\hbar}} x_1 + i\frac{p_1}{\sqrt{2m\hbar\omega}} \\ a_2 &= \sqrt{\frac{m\omega}{2\hbar}} x_2 + i\frac{p_2}{\sqrt{2m\hbar\omega}} \end{aligned}$$

and their adjoints. In particular, the first excited state of energy $E = 2\hbar\omega$ is doubly degenerate. A perturbation $V = g x_1 x_2$ is applied. Calculate the shifts of energy for the first excited state, to the lowest nontrivial order in the perturbation.

41. We consider an electron bound as an isotropic three-dimensional harmonic oscillator with

$$\mathcal{H}_0 = \frac{p_1^2}{2m} + \frac{m\omega^2 x_1^2}{2} + \frac{p_2^2}{2m} + \frac{m\omega^2 x_2^2}{2} + \frac{p_3^2}{2m} + \frac{m\omega^2 x_3^2}{2} \tag{34}$$

We can construct eigenstates using

$$\begin{aligned} a_1 &= \sqrt{\frac{m\omega}{2\hbar}} x_1 + i\frac{p_1}{\sqrt{2m\hbar\omega}} \\ a_2 &= \sqrt{\frac{m\omega}{2\hbar}} x_2 + i\frac{p_2}{\sqrt{2m\hbar\omega}} \\ a_3 &= \sqrt{\frac{m\omega}{2\hbar}} x_3 + i\frac{p_3}{\sqrt{2m\hbar\omega}} \end{aligned}$$

and their adjoints. For example, the ground state has energy $E_0 = \frac{3}{2}\hbar\omega$ and is nondegenerate and the first excited state has the energy $E_1 = \frac{5}{2}\hbar\omega$ and has degeneracy 3. Right circularly polarized light, corresponding to the polarization vector $e = (e^{(1)} + ie^{(2)})/\sqrt{2}$, is incident along the z-axis. Calculate, using first order time-dependent perturbation theory, the absorption cross section in the electric dipole approximation. What are the selection rules for the transition ?

42. A W-boson is a spin-1 particle, i.e., its spin has a j-value equal to 1. Calculate the spin matrices for this case by evaluating the matrix elements of J_i.

43. We consider a two-particle system concentrating only on the spin degrees of freedom. The spin operators for the two particles can be represented as $S_i^{(1)}$ and $S_i^{(2)}$. Each set of spins obeys the standard commutation rules and $S_i^{(1)}$ and $S_i^{(2)}$ commute with each other. The Hamitonian of the system is given to be

$$\mathcal{H} = -\lambda S_i^{(1)} S_i^{(2)} \tag{35}$$

where λ is a constant and the index i is summed over as usual. Determine the time derivatives of the matrix elements $\langle\alpha|S_i^{(1)}|\beta\rangle$ and $\langle\alpha|S_i^{(2)}|\beta\rangle$, where $|\alpha\rangle,\ |\beta\rangle$ are arbitrary states. What is the time dependence of $S_i^{(1)} + S_i^{(2)}$?

44. Determine the differential and total scattering cross sections for each of the two central potentials given below, in the Born approximation.

a.

$$V(r) = \begin{cases} -V_0, & r < a \\ 0 & r \geq a \end{cases} \tag{36}$$

b.

$$V(r) = V_0 \, \exp\left(-\frac{r}{a}\right) \tag{37}$$

45. Consider a Hydrogen-like atom with a Hamiltonian of the form

$$H_0 = \frac{p^2}{2m} - \frac{Ze^2}{r} \tag{38}$$

The atomic nucleus has a finite radius as opposed to the point-like nature assumed in the above expression. The change in the electrostatic potential due to the finite size may be written as

$$V(r) = \begin{cases} -\frac{Ze^2}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right) + \frac{Ze^2}{r} & 0 \leq r \leq R \\ 0 & r \geq R \end{cases} \tag{39}$$

Calculate the correction to the ground state energy, treating this as a perturbation, to first order.

46. Consider a charged particle in a harmonic oscillator potential. An electromagnetic wave is incident on the system. Calculate the absorption cross section with the particle making a transition from state $|n\rangle$ to state $|r\rangle$. (You must evaluate all the required matrix elements.)

47. For the previous problem, using the matrix elements you have calculated, check explicitly the Thomas-Reiche-Kuhn sum rule, viz.,

$$\sum_r (E_r - E_n)|x_{nr}|^2 = \frac{\hbar^2}{2m} \tag{40}$$

48. A spin-$\frac{1}{2}$ particle is an orbital state of orbital angular momentum $l = 1$. (Example would be the electron in the $n = 1, l = 1$ state of the Hydrogen atom.) Defining $\vec{J} = \vec{L} + \vec{S}$, find the angular momentum states which are eigenstates of J^2, J_3. In other words, work out all the relevant Clebsch-Gordan coefficients.

49. The first relativistic correction to the Hydrogen atom is given by

$$H' = -\frac{1}{8m^3c^2}\,(p_i p_i)^2 \tag{41}$$

Treating this as a perturbation, obtain the correction to the ground state energy of Hydrogen to first order in H'. (Keep in mind that you must use the Laplacian in converting $p_i p_i$ to spherical coordinates.)

50. The ammonia molecule (NH_3) can be idealized as a two-state system, correspoding to the two possible positions of the N-atom relative to the 3 H-atoms. This can be described by the matrix Hamiltonian

$$H = \begin{pmatrix} \epsilon & A \\ A & \epsilon \end{pmatrix} \tag{42}$$

a) Find the eigenstates of the system and the energy eigenvalues.

b) An electromagnetic wave is now incident on the molecule, the corresponding perturbation is given by

$$H' = \begin{pmatrix} \mu\mathcal{E}e^{-i\omega t} & 0 \\ 0 & -\mu\mathcal{E}e^{-i\omega t} \end{pmatrix} \tag{43}$$

If the system is in the lower eigenstate (found in part a)) at time $t = 0$, obtain the probability of it being in the upper state at time t, to first order in time-dependent perturbation theory.

51. A spin-1 and a spin-$\frac{1}{2}$ particle fuse together to form a bound state.

a) What are the possible spin values of the bound state?

b) What is the j-value of the state

$$|\alpha\rangle = |1, \tfrac{1}{2}, 1, \tfrac{1}{2}\rangle \tag{44}$$

where we use the notation $|j_1, j_2, m_1, m_2\rangle$. Obtain the other states with the same j-value by using $J_\pm$ operators.

52. a) Using a variational ansatz of the form $\psi = Ce^{-ar}$ where a is the variational parameter, estimate the ground state energy for a three-dimensional particle in the potential

$$V(r) = \lambda r^3 \tag{45}$$

b) Consider a potential well of the form

$$V(r) = -Ar^2 e^{-br} \tag{46}$$

where A, b are positive constants. Write down a suitable variational ansatz for the ground state wave function.

53. Consider a charged plane rotator (in the (x, y)-plane) with the Hamiltonian

$$H_0 = \frac{\hbar^2}{2K}\left(-\frac{\partial^2}{\partial\varphi^2}\right) \tag{47}$$

where φ is the azimuthal angle in the plane and the constant K is the moment of inertia.
a) Determine the eigenvalues and eigenfunctions of H.
b) An electric field is applied in the x-direction giving a perturbation

$$H_{int} = eEx = eEr\cos\varphi \tag{48}$$

Calculate the correction to the energy levels, to second order in the perturbation.

54. Consider a particle in three dimensions with the Hamiltonian

$$\begin{aligned} H &= \frac{\vec{p}\cdot\vec{p}}{2m} + V(r) \\ V(r) &= \begin{cases} V_0(\frac{r}{a} - 1), & r \le a \\ 0, & r > a \end{cases} \end{aligned} \tag{49}$$

Obtain the variational estimate of the ground state energy using a trial wavefunction of the form

$$\Psi = Ae^{-Br} \tag{50}$$

where A, B are constants.

55. Two of the lowest eigenfunctions for a Hydrogen atom are given by

$$\begin{aligned} \Psi_0 &= \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \\ \Psi_1 &= \frac{1}{\sqrt{32\pi a^3}}\left(2 - \frac{r}{a}\right) e^{-r/2a} \end{aligned}$$

Consider the ion H^- which has two electrons.

a) What are the possible eigenfunctions for the two-electron system taking account of spin and the Pauli exclusion principle ? (Electrons are spin-$\frac{1}{2}$ particles.)

b) A perturbation due to a constant (uniform) magnetic field $\vec{B} = (0, 0, B)$ gives an additional term in the Hamiltonian

$$H_{int} = \frac{eB}{m}(S^{(1)} + S^{(2)})_3 \tag{51}$$

Calculate the matrix elements of this interaction term for the states you have obtained.

56. A quantum mechanical system has a 2-dimensional Hilbert space, with energy eigenstates $|1\rangle, |2\rangle$ and corresponding unperturbed eigenvalues E_1, E_2. A perturbation of the form

$$H_{int} = \begin{pmatrix} 0 & \gamma\cos\omega t \\ \gamma\cos\omega t & 0 \end{pmatrix} \tag{52}$$

is applied. If the system is in state $|1\rangle$ at time $t = 0$, calculate the probability that it makes a transition to $|2\rangle$ in time T, using first order *time-dependent perturbation theory.*

57. Two spin-1 particles fuse together to form a single particle.

a) Neglecting degrees of freedom other than spin, what are the possible spin values of the composite particle?

b) Calculate the j-value for the state (given in the notation $|j_1, j_2, m_1, m_2\rangle$)

$$\Psi = \frac{1}{\sqrt{2}}\left(|1, 1, 1, 0\rangle - |1, 1, 0, 1\rangle\right) \tag{53}$$

and then obtain the other states with the same j-values by using the J_+, J_- operators.

58. A beam of electrons is prepared in a spin-up $S_3 = \frac{1}{2}$ state. It is then allowed to pass through a uniform magnetic field oriented in the x_1-direction, i.e., $\vec{B} = (B, 0, 0)$ for time T.

a) What is the probability of finding the value $S_1 = \frac{1}{2}$ upon a measurement ?

b) What is the probability of finding the value $S_2 = \frac{1}{2}$ upon a measurement ?

59. A particle of charge e is in the ground state of a one-dimensional harmonic oscillator at time $t = -\infty$. A uniform time-varying electric field

$$\mathcal{E} = \frac{E\tau}{e\pi}\frac{x}{t^2 + \tau^2} \tag{54}$$

in the x-direction is now applied. Calculate the probability of transition to any other energy level by time $t \to \infty$, using first order time-dependent perturbation theory. (Integrals you may encounter can be done easily by contour integration.)

60. A Galilean transformation corresponding to a change of frame of reference is expressed in canonical variables by

$$\begin{aligned} x_i \to x_i' &= x_i - \epsilon_i t \\ p_i \to p_i' &= p_i + m\epsilon_i \end{aligned}$$

where ϵ_i is the relative velocity of the two frames of reference, taken to be *infinitesimal* here. Representing this as a unitary transformation $A \to A' = UAU^\dagger$, calculate U.

61. Charmonium is a bound state of a charm quark and a charm antiquark which can be treated as a Schrödinger problem with the mass being replaced by the reduced mass $\mu = \frac{1}{2}m_c$. The binding potential is

$$V(r) = -\frac{\kappa}{r} + \sigma r \tag{55}$$

for fixed parameters κ and σ. Using a variational ansatz of the form $\psi = C\exp(-\lambda r)$, obtain the variational estimate of the ground state energy of charmonium.

62. The rotations of a nucleus are described by the anisotropic rotor Hamiltonian

$$H_0 = \frac{J_1^2 + J_2^2}{2I_1} + \frac{J_3^2}{2I_3} \tag{56}$$

where J_i are the angular momentum operators and I_1, I_3 are the moments of inertia, which are fixed numbers, with $I_1 \neq I_3$.

a) Obtain the eigenstates and eigenvalues of the Hamiltonian

b) A perturbation is now introduced, given by $H' = cJ_1$, where c is a constant. Obtain the corrected energy eigenvalues to second order in the perturbation.

63. Consider the Hamiltonian for a charged particle in a uniform magnetic field B given by

$$H = \frac{\Pi_1^2}{2m} + \frac{\Pi_2^2}{2m}$$

where $\Pi_1 = p_1 + (e/2c)Bx_2$, $\Pi_2 = p_2 - (e/2c)Bx_1$.

a) Calculate the commutator between Π_1 and Π_2. Calculate also the equations of motion for x_1, x_2, Π_1, Π_2 using the Heisenberg equations of motion (in the Heisenberg picture).

b) We also define $\tilde{\Pi}_1 = p_1 - (e/2c)Bx_2$, $\tilde{\Pi}_2 = p_2 + (e/2c)Bx_1$. (Notice the change in signs compared to Π_1, Π_2.) Calculate the commutators $[\tilde{\Pi}_1, \Pi_1]$ and $[\tilde{\Pi}_1, \Pi_2]$ and obtain the Heisenberg equations of motion for $\tilde{\Pi}_1$.

64. Obtain the differential scattering cross section in the Born approximation for the following spherically symmetric potential.

$$V(r) = \begin{cases} \frac{V_0}{r}\cos(\pi r/2R) & r \leq R \\ 0 & r \geq R \end{cases}$$

65. The Hamiltonian for a relativistic particle is given by $H = \sqrt{c^2p^2 + m^2c^4}$. A Lorentz transformation for very small velocities is given by

$$p'_i \approx p_i - v_i \frac{H}{c^2}, \qquad H' \approx H - v_i\, p_i$$

where the velocity v_i can be taken to be infinitesimal.

a) Obtain the operator K_i which generates this transformation, i.e.,

$$i\hbar\, \delta\xi = [v \cdot K, \xi]$$

for any operator ξ.

b) Obtain the commutator $[K_i, K_j]$ using your answer for K_i from part a). Identify the operator given by this commutator.

66. The Hamiltonian for a Heisenberg ferromagnet is, keeping only the nearest neighbor interactions,

$$H = -2\, J \sum_{\alpha} S^i_\alpha S^i_{\alpha+1}$$

where α labels the lattice site and $i = 1, 2, 3$, give the components of the vector operator for spin. Take an infinite one-dimensional chain and obtain the equation of motion for $\langle\psi|S^i_\alpha|\psi\rangle$.