V1100: ANALYTICAL MECHANICS

Final Examination

December 16, 2020, 11:45 AM to 2:00 PM

Answers to be returned by 2:10 PM

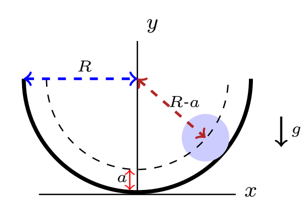
This is a closed book exam. You are supposed to know elementary trigonometric formulae and the exponential and logarithmic functions. I have also appended a list of useful formulae. These are adequate to solve all problems in this exam.

There are 5 questions. Full credit will correspond to the correct answers for the first 4 problems, for a total of 44 points. (This will be scaled appropriately to reflect the correct weight for this exam in your final grade.) Problem 5 will be treated as a bonus problem. If you work it out, it can help you make up points you may have lost elsewhere.

When you finish, please scan and send me a single pdf file of the exam, indicating on page 1 how many pages you are sending, so I will know if anything is missing in transmission. I expect the answers back by 2:10 PM at the latest. Answers returned after 2:10 PM will not be considered.

Problem 1 (*12 points*)

Consider a rail bent into a semicircular shape, of radius R, and placed on the ground as shown in figure. A disk of mass M and radius a can roll without slipping on the inside of this rail. (The motion is effectively one dimensional.)



a) Obtain the Lagrangian for the motion of the disk. (Moment of inertia of disk is $\frac{1}{2}Ma^2$.)

b) Obtain the Hamiltonian and the canonical equations of motion. (*Caution*: The distance traced out by the center of mass of the disk is different from the distance it rolls because a is not negligible.)

c) Determine the frequency for small oscillations around the equilibrium point if $a = R/3$.

Problem 2 (*10 points*)

A particle of mass m moves in three dimensions in a central potential as described by the Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} - \frac{V_0}{(a+r)^2}$$

where V_0, a are positive constants.

a) Write down the Hamilton-Jacobi equation and solve it by separation of variables, expressing your answer in terms of an integral over r. (This integral is not easy to do, so you do not have to do it at this stage.)

b) Now consider the special case of purely radial motion, setting angular motion to zero. Solve for the trajectory of the particle both for $E > 0$ and $E < 0$. (*Hint*: The substitution $u = \frac{1}{2}(a+r)^2$

will reduce the integrals you encounter to integrals of elementary functions.)

Problem 3 (*12 points*)

Consider the dynamics of a particle of mass m in three dimensions. We define the quantities

$$D_1 = \tfrac{1}{2}p^2, \quad D_2 = \tfrac{1}{2}x^2, \quad D_3 = \vec{x} \cdot \vec{p}$$

a) Calculate the Poisson brackets $\{D_i, D_j\}$ and show that they form a closed algebra.
b) Now consider the Hamitlonian

$$H = \frac{p^2}{2m} - \frac{\kappa}{r^2}$$

Consider the time-evolution of D_3 as defined by Poisson bracket with H and solve this equation for the case of a trajectory of fixed energy E.

Problem 4 (*10 points*)

a) Consider the radiation of a photon (viewed as a particle of zero mass) by a charged particle of mass m (such as an electron) moving with a constant velocity. Show by conservation of energy and momentum, that this process is forbidden in the vacuum.
b) The Lorentz transformation of the momentum of a particle of mass m is given, for small values of the velocity change for the frame of reference (denoted by ϵ_i), as

$$p_0' \approx p_0 + \epsilon_i p_i, \qquad p_i' \approx p_i + \epsilon_i p_0, \qquad p_0 = \sqrt{p^2 + m^2c^2}$$

Show that the changes in the momentum components can be given by their Poisson bracket with $-\epsilon_i K_i$ with $K_i = x_i \sqrt{p^2 + m^2c^2}$.

Problem 5 (Bonus, *8 points*)

The Lagrangian describing the motion of a spinning top placed on the ground is given in terms of the three Euler angles as

$$L = \tfrac{1}{2}I_1 \left(\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta\right) + \tfrac{1}{2}I_3 \left(\dot{\varphi} + \dot{\psi}\cos\theta\right)^2 - Mgh\cos\theta$$

Obtain the canonical momenta and the Hamiltonian (in terms of the coordinates and canonical momenta).

Useful results
General

$$\begin{aligned}
ds^2 &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 && \text{(Cartesian coordinates)}\\
&= dr^2 + r^2 d\varphi^2 + dz^2 && \text{(Cylindrical coordinates)}\\
&= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2 && \text{(Spherical polar coordinates)}\\
L &= T - V, \qquad T = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2), && \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}
\end{aligned}$$

Rotating frame, Euler equations

$$L = \frac{m}{2}\left(\dot{x}^2 - 2\,\epsilon_{ijk}\dot{x}^i x^j \omega^k + \omega^2 x^2 - (\vec{\omega}\cdot\vec{x})^2\right) - V(x)$$

$$\begin{aligned} I_1\dot{\Omega}_1 + (I_3 - I_2)\Omega_2\Omega_3 &= \tau_1 \\ I_2\dot{\Omega}_2 + (I_1 - I_3)\Omega_3\Omega_1 &= \tau_2 \\ I_3\dot{\Omega}_3 + (I_2 - I_1)\Omega_1\Omega_2 &= \tau_2 \end{aligned}$$

Moment of inertia

$$I = \frac{2}{5}MR^2 \quad \text{(Sphere)}, \qquad I = \frac{1}{2}MR^2 \quad \text{(Disk around its axis)}$$

Canonical formulation

$$\begin{aligned} p_i &= \frac{\partial L}{\partial \dot{q}_i}, \qquad H(p,q) = p_i\dot{q}_i - L \\ \dot{q}_i &= \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad \frac{\partial f}{\partial t} = \{f, H\} \\ \{f, h\} &= \sum_k \left(\frac{\partial f}{\partial q^k}\frac{\partial h}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial h}{\partial q^k}\right) \end{aligned}$$

Relativity

4-momentum $p_\mu = (p_0, p_i)$, $p_0 = H/c$.

$$\eta_{\mu\nu}p^\mu p^\nu = p_0^2 - \vec{p}\cdot\vec{p} = m^2c^2, \qquad p_0 = \sqrt{\vec{p}^2 + m^2c^2}$$