V1100: ANALYTICAL MECHANICS

Midterm Examination I

October 19, 2020, 11:45 AM to 1:25 PM

Answers to be returned by 1:35 PM

This is a closed book exam. You are supposed to know elementary trigonometric formulae and the exponential and logarithmic functions. I have also appended a list of useful formulae. These are adequate to solve all problems in this exam.

There are 4 questions. Full credit will correspond to the correct answers for the first 3 problems, for a total of 35 points. (This will be scaled appropriately to reflect the correct weight for this exam in your final grade.) Problem 4 will be treated as a bonus problem. If you work it out, it can help you make up points you may have lost elsewhere.

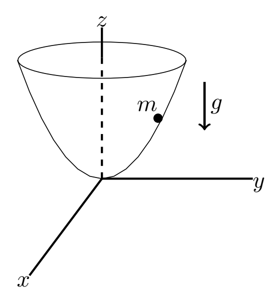
When you finish, please scan and send me a single pdf file of the exam, indicating on page 1 how many pages you are sending, so I will know if anything is missing in transmission. I expect the answers back by 1:35 PM at the latest. Answers returned after 1:35 PM will not be considered.

Problem 1 (*15 points*)

A bead of mass m can slide on the inside surface of a parabolic dish with negligible friction. The dish can be described by the equation $z = \frac{1}{2}\lambda(x^2 + y^2)$, λ being a positive real constant. The dish is placed vertically on the ground as shown in figure.

a) Find the Lagrangian and the equations of motion for the bead, using cylindrical coordinates. (Keep in mind that there is angular motion as well.)

b) Find the value of $r = \sqrt{x^2 + y^2}$ for which a circular orbit is possible.



Problem 1

Solution

The position of the bead is given by (x, y, z) with $z = \frac{1}{2}\lambda(x^2 + y^2)$. In cylindrical coordinates, we get

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad z = \tfrac{1}{2}\lambda\, r^2$$

$$\dot{x} = \dot{r}\cos\varphi - r\sin\varphi\,\dot{\varphi}, \qquad \dot{y} = \dot{r}\sin\varphi + r\cos\varphi\,\dot{\varphi}, \qquad \dot{z} = \lambda r\dot{r}$$

Thus

$$\begin{aligned} T &= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2 + \lambda^2 r^2\dot{r}^2\right) \\ &= \tfrac{1}{2}m\left((1 + \lambda^2 r^2)\dot{r}^2 + r^2\dot{\varphi}^2\right) \end{aligned}$$

The potential energy is due to the force of gravity and is given by

$$V = mgz = \tfrac{1}{2}mg\lambda r^2$$

a) The Lagrangian is thus

$$L = \tfrac{1}{2}m\left((1+\lambda^2 r^2)\,\dot{r}^2 + r^2\dot{\varphi}^2\right) - \tfrac{1}{2}mg\lambda r^2$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{r}} &= m(1+\lambda^2 r^2)\,\dot{r}, \qquad \frac{\partial L}{\partial r} = m\left[\lambda^2\, r\dot{r}^2 + r\dot{\varphi}^2 - g\lambda r\right] \\
\frac{\partial L}{\partial \dot{\varphi}} &= mr^2\dot{\varphi}, \qquad \frac{\partial L}{\partial \varphi} = 0
\end{aligned}$$

The equations of motion are thus given by

$$\begin{aligned}
(1+\lambda^2 r^2)\ddot{r} + \lambda^2 r\dot{r}^2 &= r\dot{\varphi}^2 - g\lambda\, r \\
\frac{d}{dt}(mr^2\dot{\varphi}) &= 0
\end{aligned}$$

b) $mr^2\dot{\varphi} = l$ is a constant of motion. From the equation of motion for r, we see that we can have a circular orbit with $\dot{r}=0$, $\ddot{r}=0$, if

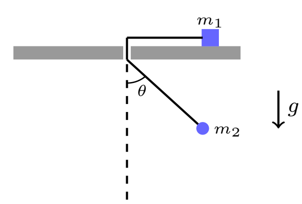
$$r\dot{\varphi}^2 - g\lambda r = \frac{l^2}{m^2 r^3} - g\lambda r = 0$$

This gives the value of r for which a circular orbit is possible as

$$R = \left(\frac{l^2}{m^2 g\lambda}\right)^{\frac{1}{4}}$$

<u>Problem 2</u> (*10 points*)

A mass m_1 can slide without friction on a horizontal table. A string of negligible mass is attached to it and passes down through a hole in the center of the table. A mass m_2 is attached to the other end of the string and can move as a pendulum (of variable length since m_1 can slide). For simplicity, you can assume that the entire motion is in a single vertical plane. Obtain the Lagrangian and the equations of motion for the masses. Keep in mind that the total length of the string is fixed. (Ignore the thickness of the tabletop, although I magnified it for pictorial clarity.)



Problem 2

<u>Solution</u>

Take the motion to be in the (x, y)-plane and let $(X, 0)$ be the position of the mass m_1. Also let L denote the length of the whole string, while s denotes length of the string beneath the tabletop. For the position of the mass m_2, we have

$$x = s\,\sin\theta, \qquad y = -s\,\cos\theta$$

Further $X + s = L$, with L fixed. Thus

$$\dot{X} = -\dot{s}, \qquad \dot{x} = \dot{s}\sin\theta + s\dot{\theta}\,\cos\theta, \qquad \dot{x} = -\dot{s}\cos\theta + s\dot{\theta}\,\sin\theta$$

Thus

$$\begin{aligned} T &= \tfrac{1}{2}m_1\dot{X}^2 + \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) \\ &= \tfrac{1}{2}(m_1+m_2)\dot{s}^2 + \tfrac{1}{2}m_2 s^2\dot{\theta}^2 \end{aligned}$$

The potential energy is $V = m_2 g y = -m_2 g s\cos\theta$. This leads to

$$L = \tfrac{1}{2}(m_1+m_2)\dot{s}^2 + \tfrac{1}{2}m_2 s^2\dot{\theta}^2 + m_2 g s\cos\theta$$

$$\begin{aligned} \frac{\partial L}{\partial \dot{s}} &= (m_1+m_2)\dot{s}, \qquad \frac{\partial L}{\partial s} = m_2\, s\,\dot{\theta}^2 + m_2 g\cos\theta \\ \frac{\partial L}{\partial \dot{\theta}} &= m_2 s^2\dot{\theta}, \qquad\quad \frac{\partial L}{\partial \theta} = -m_2 g s\sin\theta \end{aligned}$$

The equations of motion are then given by

$$\begin{aligned} \ddot{s} &= \frac{m_2}{m_1+m_2}\left[s\,\dot{\theta}^2 + g\,\cos\theta\right] \\ \frac{d}{dt}(s^2\,\dot{\theta}) &= -g\,s\,\sin\theta \end{aligned}$$

Problem 3 (*10 points*)

A particle of mass m can move in three dimensions under the influence of a central potential $V(r) = \sigma\, r$, where σ is a positive constant. (Such a potential governs the bound states of a charm quark with its antiquark.)

a) Show that there should be a minimum value E_{min} for the energy. Find E_{min} as a function of σ, m and the angular momentum l.

b) For what value of the radius is a circular orbit possible?

c) The particle in this circular orbit is perturbed by another particle passing by, which changes R to $R+\eta(t)$. Write down the equation of motion for η to first order in η. (Higher order terms in η can be neglected.) Solve this equation to show that the particle oscillates in the radius as it goes around the center.

Solution

a) Because we have a central potential, we can take the motion to be confined to a plane by virtue of the conservation of the direction of angular momentum. Thus the kinetic and potential energies are

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2), \qquad V = \sigma\, r$$

The conservation of the magnitude of angular momentum gives

$$m r^2\dot{\varphi} = l = \text{constant of motion}$$

Using this, the expression for the energy is given by

$$E = T + V = \tfrac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + \sigma r \equiv \tfrac{1}{2}m\dot{r}^2 + U_{\text{eff}}$$

Since $\dot{r}^2$ has to be positive, we get

$$E \geq U_{\text{eff}}$$

Thus the minimum possible value for E is the lowest possible value for U_{eff}. We find this by setting the derivative of U_{eff} to zero.

$$-\frac{l^2}{mr^3} + \sigma = 0 \implies r = \left(\frac{l^2}{m\sigma}\right)^{\frac{1}{3}} \equiv R$$

The value of U_{eff} at this point is

$$U_{\text{eff}}|_{\text{min}} = \frac{3}{2}\left(\frac{l^2\sigma^2}{m}\right)^{\frac{1}{3}} = E_{\text{min}}$$

b) The equation for r is given by

$$m\ddot{r} = \frac{l^2}{mr^3} - \sigma$$

For a circular orbit, we need $\ddot{r} = 0$. The right hand side must thus vanish. The value of r obeying this condition is the same as what minimizes U_{eff}. Thus circular orbit is possible for a radius r given by

$$R = \left(\frac{l^2}{m\sigma}\right)^{\frac{1}{3}}$$

c) We write $r = R + \eta$ for the perturbed orbit. Substituting in the equation of motion for r, we get

$$\begin{aligned} m\ddot{\eta} &= \frac{l^2}{m(R+\eta)^3} - \sigma = \frac{l^2}{mR^3} - \sigma - \frac{3l^2}{mR^4}\,\eta + \mathcal{O}(\eta^2) \\ &= -\frac{3\,l^2}{mR^4}\,\eta + \mathcal{O}(\eta^2) \end{aligned} \tag{1}$$

We used the binomial theorem to expand the denominator and kept only terms up to the first power of η. The equation for η is the same as what we get for an oscillator or for a small amplitude pendulum. The solution is evidently given by

$$\eta = A\sin(\omega t + \delta)$$

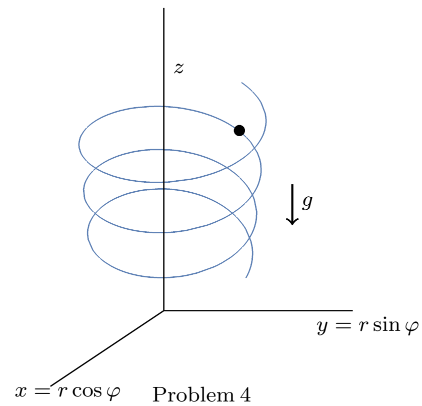
where $\omega = \sqrt{3\,l^2/mR^4}$. We see that η is oscillatory and bounded, so $r = R + \eta$ corresponds to oscillatory motion around the circular orbit.

Problem 4 (Bonus, *8 points*)

A wire is twisted into a vertical helix as shown in figure and held fixed. The helix may be described in cylindrical coordinates (r, φ, z) by $z = \lambda\varphi$, where λ is a positive constant. A bead of mass m can slide down the helix under the action of gravity. We will neglect friction for this problem.

a) Find the Lagrangian and the equations of motion for the bead.

b) Solve the equation of motion with the initial conditions $\varphi(t = 0) = 0$ and $\dot{\varphi}(t = 0) = \omega$, where ω is a constant.



Problem 4

Solution

a) The position of the bead is given by

$$(x, y, z) = (r\cos\varphi, r\sin\varphi, \lambda\varphi)$$

Taking the derivative, the velocity is given by

$$v = \dot{\varphi}\,(-r\sin\varphi, r\cos\varphi, \lambda)$$

We used the fact that the radius r of the helix is a constant since the shape of the wire is fixed. Thus we find that

$$T = \tfrac{1}{2}m\dot{\varphi}^2(r^2+\lambda^2), \qquad V = mg\lambda\varphi$$

with $L = T - V$.

$$\frac{\partial L}{\partial \dot{\varphi}} = m(r^2+\lambda^2)\,\dot{\varphi}, \qquad \frac{\partial L}{\partial \varphi} = -mg\lambda$$

The equation of motion is

$$(r^2+\lambda^2)\,\ddot{\varphi} = -g\lambda$$

b) Apart from constants this is familiar as the equation for free fall. We can directly integrate twice to get

$$\begin{aligned}\varphi(t) &= \varphi(0) + \dot{\varphi}(0)\,t - \tfrac{1}{2}\frac{g\lambda}{r^2+\lambda^2}\,t^2 \\ &= \omega\,t - \tfrac{1}{2}\frac{g\lambda}{r^2+\lambda^2}\,t^2\end{aligned}$$

where we used the given initial date in the second line.

Useful results

$$\begin{aligned}ds^2 &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 &\qquad& \text{(Cartesian coordinates)} \\ &= (dx)^2 + (dy)^2 + (dz)^2 && \text{(Cartesian coordinates)}\end{aligned}$$

$$\begin{aligned} &= dr^2 + r^2 d\varphi^2 + dz^2 \qquad &\text{(Cylindrical coordinates)} \\ &= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \qquad &\text{(Spherical polar coordinates)} \end{aligned}$$

$$L = T - V, \qquad T = \tfrac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

$$(1+x)^n = 1 + n\,x + \frac{n(n-1)}{2}x^2 + \frac{n(n-1))n-2)}{3!}x^3 + \cdots$$