PHY V25: QUANTUM MECHANICS I

Additional Practice Problems

Problem 1

a) If L_i are the components of the angular momentum operator show that

$$[L_i, p^2] = 0, \qquad [L_i, r^2] = 0$$

b) Consider the Hamiltonian

$$H = \frac{\vec{p}\cdot\vec{p}}{2m} + \frac{m\omega^2}{2}(x^2 + y^2 + z^2)$$

This corresponds to the 3-dimensional isotropic harmonic oscillator. Calculate $[L_i, H]$.
c) From the commutator in part b, what is $\langle \partial L_i/\partial t \rangle$?

Problem 2

In this problem we consider the two-dimensional harmonic oscillator. The Hamiltonian is given by

$$H = \frac{\vec{p}\cdot\vec{p}}{2m} + \frac{m\omega^2}{2}(x^2 + y^2)$$

In polar coordinates, the Schrödinger equation is given by

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{m\omega^2}{2}r^2\,\psi = E\,\psi$$

More explicitly,

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2}\right] + \frac{m\omega^2}{2}r^2\,\psi = E\,\psi$$

Carry out a separation of variables to obtain the radial and angular equations. Solve the angular equation and obtain the radial equation. Write the radial equation in terms of the variable

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\,r$$

Determine the behavior of the wave function near $r = 0$ and as $r \to \infty$.

Problem 3

In three dimensions, we consider a wave function of the form

$$\psi = Ce^{-r/a}$$

a) Determine the normalization constant C.
b) The Hamiltonian for the Hydrogen atom (with $Z = 1$) is given by

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{r}$$

Calculate the expectation value

$$\mathcal{E}(a) \equiv \langle H \rangle = \int d^3x\, \psi^*\, H\, \psi$$

as a function of a. Minimize $\mathcal{E}(a)$ with respect to a and calculate the value $\mathcal{E}(a_*)$ at the point a_*, where the minimum occurs. Compare your answer to the exact eigenvalue for the ground state of the Hydrogen atom obtained in class.

Problem 4

Here we consider a variant of the problem given above. A particle of mass μ in an *a priori* unknown central potential $V(r)$ has the Hamiltonian

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 + V(r)$$

The wave function for the ground state is known to be

$$\psi = \mathcal{C}\, e^{-r/a}$$

for some constants $a, \mathcal{C}$. The potential $V(r)$ is known to vanish as $r \to \infty$.

a) Set up the Schrödinger equation and use the large r behavior to identify the energy eigenvalue E for this state.

b) Once you know E, determine the potential $V(r)$ using using the fact that ψ satisfies the Schrödinger equation for all r.

Problem 5

The Hamiltonian for a rotating body is given by

$$H = \frac{1}{2\,I}\sum_j L_j L_j + \mu B\, L_3$$

where μ, B are constants; I is the moment of inertia, taken to be another constant.

a) Obtain the eigenvalues and eigenstates of this Hamiltonian using angular momentum theory.

b) What are the degeneracies for the energy eigenstates?

c) Obtain the time-evolution of $\langle L_i \rangle$ (for all i) using

$$i\hbar \frac{\partial \langle L_i \rangle}{\partial t} = \langle L_i H - H L_i \rangle$$

Problem 6

Here we will consider two independent harmonic oscillators using the formalism of step-up and step-down operators. Since there are two oscillators, we will consider two mutually commuting sets, a, $a^\dagger$, b, $b^\dagger$, with the commutation rules

$$[a, a^\dagger] = 1, \qquad [a, a] = [a^\dagger, a^\dagger] = 0$$

$$[b, b^\dagger] = 1, \qquad [b, b] = [b^\dagger, b^\dagger] = 0$$
$$[a, b] = [a^\dagger, b^\dagger] = [a, b^\dagger] = [a^\dagger, b] = 0$$

Define the operators

$$J_1 = \frac{\hbar}{2}(a^\dagger b + b^\dagger a), \qquad J_2 = -i\frac{\hbar}{2}(a^\dagger b - b^\dagger a), \qquad J_3 = \frac{\hbar}{2}(a^\dagger a - b^\dagger b)$$

a) Calculate the commutation rules among the J_i and show that they reproduce the angular momentum commutation rules.
b) Calculate J^2. (You will encounter the operator $N = a^\dagger a + b^\dagger b$ in addition to combinations given before.)
c) In a way analogous to what we did for the oscillator, introduce the states

$$|n, k\rangle = \frac{(a^\dagger)^{n-k}}{\sqrt{(n-k)!}} \frac{(b^\dagger)^k}{\sqrt{k!}} |0\rangle, \qquad k = 0, 1, \cdots, n$$

(Recall that $a|0\rangle = b|0\rangle = 0$.) Show that N has the same eigenvalue on all these n states. Further, calculate J^2 and J_3 on these states. Show that these results are in one-to-one correspondence with the results on angular momentum eigenstates we derived in terms of $J_\pm$, J_3 in class.

Problem 7

a) In the theory of angular momentum, if $|\alpha\rangle$ is an eigenstate of J_3, then show that $\langle\alpha| J_1 |\alpha\rangle = 0$, $\langle\alpha| J_2 |\alpha\rangle = 0$.
b) From this result, the mean square uncertainty in the measurement of J_1 and J_2 can be taken as

$$\Delta J_1^2 = \langle J_1^2\rangle - (\langle J_1\rangle)^2 = \langle J_1^2\rangle, \qquad \Delta J_2^2 = \langle J_2^2\rangle - (\langle J_2\rangle)^2 = \langle J_2^2\rangle$$

Consider the states

$$|A\rangle = J_1 |\alpha\rangle, \qquad |B\rangle = J_2 |\alpha\rangle$$

Using the Cauchy-Schwartz inequality for these states, obtain an uncertainty principle for simultaneous measurements of J_1 and J_2. Express your answer in terms of the value J_3 for the state $|\alpha\rangle$.

Problem 8

An electron is prepared to be in the spin state given as

$$|\psi\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

This may be done using a Stern-Gerlach apparatus which is suitably oriented. Now a measurement of $S_2 = \frac{\hbar}{2}\sigma_2$ is carried out. What is the probability of obtaining the value $\frac{\hbar}{2}$? What is probability of obtaining $-\frac{\hbar}{2}$?

Problem 9

A Stern-Gerlach apparatus is set up such that it produces eigenstates of the combination $\cos\varphi\, S_1 + \sin\varphi S_2$. This is for an electron with $S_i = (\hbar/2)\sigma_i$. Obtain the eigenstates and eigenvalues of this operator. If a measurement of S_3 is now carried out, what is the probability of obtaining $(\hbar/2)$?

Problem 10

An electron is in an eigenstate of the Hamiltonian with eigenvalue E_0, which is doubly degenerate. (There are two eigenstates of the same energy.) We now put this in a magnetic field along the x-axis so that the Hamiltonian, restricted to these two states, becomes

$$H = \begin{pmatrix} E_0 & -\frac{1}{2}\mu B \\ -\frac{1}{2}\mu B & E_0 \end{pmatrix}$$

Determine the eigenvalues and eigenstates of this Hamiltonian.

Problem 11

The Hamiltonian for a particle in a central potential has the form

$$H = \frac{p^2}{2M} + V(r) + \lambda\, \vec{L}\cdot\vec{S}$$

where $\vec{L}\cdot\vec{S} = \sum_i L_i S_i$ is the scalar product of the orbital angular momentum and the spin angular momentum.

a) Calculate the rate of change with time of the expectation value of L_i and S_i, i.e., calculate

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\langle L_i\rangle &= \Big\langle L_i H - H L_i \Big\rangle \\ i\hbar\frac{\partial}{\partial t}\langle S_i\rangle &= \Big\langle S_i H - H S_i \Big\rangle \end{aligned}$$

Show that each of this gives nonzero change under time-evolution.

b) Using these results, identify the rate of change of the expectation value of the total angular momentum $\langle J_i\rangle$, where $J_i = L_i + S_i$.

Problem 12

A charged spin-$\frac{1}{2}$ particle (hence carrying nonzero magnetic moment) is in a magnetic field

$$B_1 = b\,x, \qquad B_2 = 0, \qquad B_3 = -b\,z$$

where b is a constant. With no additional potentials, the motion of the particle is described by the Hamiltonian

$$H = \frac{p^2}{2M} + \mathcal{C}\,\sigma \cdot \vec{B}$$

Calculate how the trajectory of the particle is affected by the given inhomogeneous magnetic field. This means that you should calculate

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \langle x_i \rangle &= \Big\langle x_i H - H x_i \Big\rangle \\ i\hbar \frac{\partial}{\partial t} \langle p_i \rangle &= \Big\langle p_i H - H p_i \Big\rangle \end{aligned}$$

(You can use one in the other to obtain the second derivative of $\langle x_i \rangle$.) For the state with which you take the expectation value, use the $S_3 = \hbar/2$ state and the $S_3 = -\hbar/2$ state in turn. (This should tell you how the Stern-Gerlach apparatus works.)

Problem 13

You are given the angular part of a wave function

$$\psi = \sqrt{\frac{15}{32\pi}}\,(\sin\theta)^2\, e^{2i\varphi}$$

a) Show that this is the state of highest value for L_3 for a certain choice of l. Identify l.
b) Use the action of L_- to obtain all states of lower values for L_3 for the given l.

Problem 14

There is a spherically symmetric potential (or central potential) $V(r)$ which is *a priori* unknown, except that we know that $V = b$ at $r = 0$. However, the ground state wave function is known to have the form

$$\Psi_0(r) = \mathcal{C}\, e^{-\lambda r^2}$$

for some constants $\mathcal{C}$ and λ. Set up the time-independent Schrödinger equation for this problem. Identify the potential $V(r)$ and the ground state energy E_0.