V2500: QUANTUM MECHANICS I

Midterm Examination

October 15, 2019, 10:00 to 11:40 AM

This is a closed book exam. You are supposed to know elementary trigonometric formulae and the exponential and logarithmic functions; these are adequate to solve all problems in this exam. You must answer all questions correctly for full credit.

Problem 1 (10 points)

a. If A and B are two operators, show, by expanding out both sides to the appropriate order, that

$$e^A\, e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$$

where the ellipsis denote terms with at least two As and one B, or two Bs and one A, or terms with higher powers. Recall that a function of an operator is defined by the power series expansion of the function.

b. From the basic commutation rules of quantum mechanics, calculate $x\,p^n - p^n\,x$, where $n > 2$.

Problem 2 (15 points)

A particle in a harmonic oscillator potential is prepared to be in the quantum state given by

$$\psi(x) = \left[\frac{16m\omega}{9\pi\hbar}\right]^{\frac{1}{4}} \xi^2 \exp(-\tfrac{1}{2}\xi^2)$$

a. Calculate the inner products $\langle\psi_0|\psi\rangle$, $\langle\psi_1|\psi\rangle$ and $\langle\psi_2|\psi\rangle$, where the energy eigenstates of the oscillator are given by

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}\; H_n(\xi)$$

b. If a measurement is now carried out on this system, what are the probabilities of finding the particle in the ground state ψ_0, the first excited state ψ_1 and second excited state ψ_2?

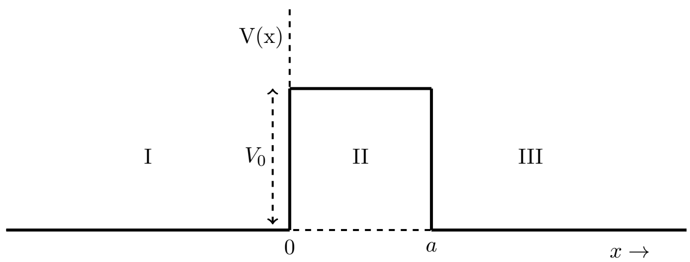
c. What is the expectation value of the energy for the state ψ?

Problem 3 (15 points)

Consider particles of energy E incident from the left on the potential barrier

$$V(x) = \begin{cases} 0 & x < 0,\ x > a \\ V_0 & 0 < x < a \end{cases}$$

This is shown in the figure. We take $E > V_0$. Calculate the reflection coefficient for this problem.



<u>USEFUL FORMULAE</u>

Basic rules

$$x_i x_j - x_j x_i = 0, \qquad p_i p_j - p_j p_i = 0$$
$$x_i p_j - p_j x_i = i\hbar\delta_{ij}$$

The inner product of two states with wave functions χ and ψ is given by

$$\langle\chi|\psi\rangle = \int dx\, \chi^* \psi$$

Oscillator

$$H\,|n\rangle = (n + \tfrac{1}{2})\hbar\omega\,|n\rangle$$
$$\langle x|n\rangle = \psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \; e^{-\frac{1}{2}\xi^2} H_n(\xi)$$
$$\xi = \sqrt{(m\omega/\hbar)}\, x$$
$$H_0(\xi) = 1, \qquad H_1(\xi) = 2\xi, \qquad H_2(\xi) = -2 + 4\xi^2,$$
$$H_3(\xi) = -12\xi + 8\xi^3, \qquad H_4(\xi) = 12 - 48\xi^2 + 16\xi^4$$
$$a = \sqrt{\frac{m\omega}{2\hbar}}\, x + i\frac{p}{\sqrt{2m\hbar\omega}}$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, x - i\frac{p}{\sqrt{2m\hbar\omega}}$$
$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle, \qquad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Integrals

$$\int_{-\infty}^{\infty} dy\, e^{-y^2} = \sqrt{\pi}, \qquad \int_{-\infty}^{\infty} dy\, y^2\, e^{-y^2} = \frac{1}{2}\sqrt{\pi}, \qquad \int_{-\infty}^{\infty} dy\, y^4\, e^{-y^2} = \frac{3}{4}\sqrt{\pi}$$