Problem Set 1

Due September 11, 2025

Problem 1

In class I calculated the elements $X_{nn'}$. In a similar way, we define

$$P_{nn'} = \int_0^L dx\, v_n(x) \left(-i\hbar \frac{\partial}{\partial x}\right) v_{n'}(x)$$

where

$$v_n(x) = \sqrt{\frac{2}{L}} \sin(n\pi x/L), \quad x \in [0, L]$$

Carry out the integration to obtain $P_{nn'}$. Also show that $P_{nn'}^* = P_{n'n}$.

Problem 2

Consider the function

$$\psi(x) = \mathcal{C}\, x\, \sin(\pi x/L)$$

which is defined on the interval $[0, L]$.

a) Identify the value of $\mathcal{C}$ such that ψ is normalized, i.e., such that

$$\int_0^L dx\, \psi^* \psi = 1$$

b) The function ψ is square-integrable over the interval $[0, L]$ and vanishes at $x = 0, L$. So it can be expanded in terms of the basis $\{v_n(x)\}$ given in Problem 1 as

$$\psi(x) = \sum_n c_n\, v_n(x)$$

Calculate the coefficients c_n.

Problem 3

Consider a Hilbert space and a basis of vectors $\{|e_i\rangle\}$, $i = 1, 2, 3, \cdots$. Unlike what we did in class, we will not assume that these are orthonormal. Thus

$$\langle e_i | e_j \rangle = g_{ij}$$

where g_{ij} is not δ_{ij}, but can be taken to be invertible.

a) Show that the matrix $\mathbf{g}$ (with matrix elements g_{ij}) is hermitian.

b) Expand an arbitrary vector in terms of this basis as

$$|a\rangle = \sum_i a_i\, |e_i\rangle$$

Calculate the coefficients a_i in terms of the inner product $\langle e_i|a\rangle$.

c) Using the solution for part b), obtain the completeness relation for the basis $\{|e_i\rangle\}$.

Problem 4

We have considered the basis $\{v_n\}$ as in Problem 1. Now consider a different set of functions defined on the interval $[0, L]$ as

$$u_n(x) = \sqrt{\frac{2}{L}}\cos(n\pi x/L), \quad n = 1, 2, \cdots .$$

These functions do not vanish at the boundaries. Rather they obey $u'_n(x) = 0$ at $x = 0, L$. (These are what are called Neumann boundary conditions.) Carry out the needed integration to identify the integral

$$\int_0^L dx\, u_n(x)\, u_{n'}(x)$$