PHY V2500: QUANTUM MECHANICS I

Problem Set 3

Due October 7, 2025

Problem 1

In the discussion of the harmonic oscillator in class, I introduced the operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}, \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$$

I also showed that the eigenstates of the Hamiltonian are given by

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n\,|0\rangle$$

a) Writing $\hat{x}$ and $\hat{p}$ in terms of a and $a^\dagger$, obtain the matrix elements of the momentum operator defined by

$$P_{mn} = \langle m|\,\hat{p}\,|n\rangle$$

(*Do not try to write an infinite-dimensional matrix; use the Kronecker delta to make the notation compact.*)

b) We define the operators

$$R_+ = \tfrac{1}{2}(a^\dagger)^2, \qquad R_- = \tfrac{1}{2}(a)^2, \qquad R_3 = a^\dagger a + \tfrac{1}{2}$$

Work out the commutation rules $[R_+, R_-]$, $[R_3, R_\pm]$.

c) Calculate the matrix elements $\langle n|\,R_\pm\,|m\rangle$.

Problem 2

In this problem you will check orthonormality of two of the energy eigenstates for the oscillator using the explicit formulae for the wave functions. Consider the first and second excited states of the oscillator given by the wave functions

$$\begin{aligned}
\psi_1(x) = \langle x|1\rangle &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2}}H_1(\xi)\,e^{-\frac{1}{2}\xi^2} \\
\psi_2(x) = \langle x|2\rangle &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{8}}H_2(\xi)\,e^{-\frac{1}{2}\xi^2} \\
H_1(\xi) = 2\xi, &\qquad H_2(\xi) = 4\xi^2 - 2
\end{aligned}$$

where $\xi = \sqrt{m\omega/\hbar}\;x$.

a) Calculate $\langle 1|1\rangle$, $\langle 2|2\rangle$, $\langle 1|2\rangle$ using the definition of the inner product

$$\langle n|m\rangle = \int_{-\infty}^{\infty} dx\,\psi_n^*\,\psi_m$$

(*You must show how you worked out the integrals.*)

Problem 3

Consider the harmonic oscillator with the eigenfunctions ψ_n of the Hamiltonian as given in class. We take a state at time $t = 0$ given by

$$\psi = A(\psi_0 + \psi_1)$$

a) Find the normalization factor A.

b) Consider the wave function at time t and calculate $\langle x \rangle$ and $\langle p \rangle$ at time t using these wave functions. (Your answers will have some time dependence.)

c) Show that for *every odd value of* n, the eigenfunctions $\psi_n(x)$ of the Hamiltonian for the oscillator vanish at $x = 0$.

Problem 4

Consider a particle which can move in one dimension, but on the *half-line* $0 \leq x \leq \infty$. Write down the condition for hermiticity of the momentum operator. What is the boundary condition on the wave functions at $x = 0$ to ensure that the momentum operator has the right hermiticity property?