Problem Set 5

Due November 11, 2025

Problem 1

In class I derived the formulae for the action of $J_\pm$ on the angular momentum eigenstates $|j, m\rangle$,

$$J_+ |j, m\rangle = \hbar\sqrt{j(j+1) - m(m+1)} \, |j, m+1\rangle$$
$$J_- |j, m\rangle = \hbar\sqrt{j(j+1) - m(m-1)} \, |j, m-1\rangle$$

where $J_\pm = J_1 \pm iJ_2$. You will find these formulae in my notes as well. Recall also that

$$J_3 |j, m\rangle = m\hbar \, |j, m\rangle$$

a) Use these formulae to calculate J_1, J_2, J_3 as matrices for $j = 1$. (I worked out the case of $j = \frac{1}{2}$ in class obtaining the Pauli matrices.) In the present case, you should get 3×3 matrices.

b) Use them to calculate the commutators $[J_i, J_j]$ for all i, j. Use matrix multiplication, not operator identities.

Solution

a) For $j = 1$, we have three possible states corresponding to $m = 1, 0, -1$. From the given formula

$$J_+ |1, 1\rangle = 0, \quad J_+ |1, 0\rangle = \hbar\sqrt{2} \, |1, 1\rangle, \quad J_+ |1, -1\rangle = \hbar\sqrt{2} \, |1, 0\rangle$$

The matrix elements are of the form $\langle 1, m' | J | 1, m \rangle$. Thus

$$J_3 = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad J_+ = \hbar \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

J_- is the hermitian conjugate of J_+, so we can easily write it down as

$$J_- = \hbar \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

We the find

$$J_1 = \tfrac{1}{2}(J_+ + J_-) = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_- = -\tfrac{i}{2}(J_+ - J_-) = -i\frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

b) By matrix multiplication

$$J_3 J_1 = \frac{\hbar^2}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{\hbar^2}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$J_1 J_3 = (J_3 J_1)^\dagger == \frac{\hbar^2}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus

$$[J_3, J_1] = \frac{\hbar^2}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = i\hbar J_2$$

Similarly,

$$J_1 J_2 = \frac{\hbar^2}{2i} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad J_2 J_1 = (J_1 J_2)^\dagger = -\frac{\hbar^2}{2i} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Thus

$$[J_1, J_2] = \frac{\hbar^2}{2i} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = i\hbar J_3$$

The last relation $[J_2, J_3] = i\hbar J_1$ follows the same way.

Problem 2

The orbital angular momentum operator is defined as

$$\vec{L} = \vec{x} \times \vec{p}, \implies L_1 = x_2 p_3 - x_3 p_2, \ L_2 = x_3 p_1 - x_1 p_3, \ L_3 = x_1 p_2 - x_2 p_1$$

Using this definition and the Heisenberg algebra, calculate the commutators $[L_i, x_j]$, $[L_i, p_j]$, for all $i, j = 1, 2, 3$. Calculate also $[L_i, x_1^2 + x_2^2 + x_3^2]$.

Solution

The angular momentum operator is $L_i = \sum_{b,c} \epsilon_{ibc} x_b p_c$. Thus

$$[L_i, x_j] = \sum_{b,c} \epsilon_{ibc} [x_b p_c, x_j] = \sum_{b,c} \epsilon_{ibc} x_b [p_c, x_j] == -i\hbar \sum_{b,c} \epsilon_{ibc} x_b \, \delta_{cj} = i\hbar \sum_b \epsilon_{ijb} x_b$$

$$[L_i, p_j] = \sum_{b,c} \epsilon_{ibc} [x_b p_c, p_j] = \sum_{b,c} \epsilon_{ibc} [x_b, p_j] p_c == i\hbar \sum_{b,c} \epsilon_{ibc} \delta_{bj} p_c = i\hbar \sum_c \epsilon_{ijc} p_c$$

$$[L_i, x_1^2 + x_2^2 + x_3^2] = \sum_j [L_i, x_j x_j] = \sum_j \Big([L_i, x_j] x_j + x_j [L_i, x_j] \Big)$$
$$= i\hbar \sum_{j,k} \epsilon_{ijk} \big(x_k x_j + x_j x_k \big) = 0$$

The last equality follows form the fact that ϵ_{ijk} is antisymmetric in j, k and the bracketed terms are symmetric in j, k, so various terms cancel out upon summation. Sincs the commutator with the momentum has a similar form, we also have

$$[L_i, p_1^2 + p_2^2 + p_3^2] = 0$$

Problem 3

We consider a two-particle system in one dimension with the Hamiltonian

$$H = -\frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2} \frac{\partial^2}{\partial x_2^2} + V(x_1 - x_2)$$

x_1 and x_2 refer to the particles labeled 1 and 2 and likewise for the masses. I have written this as a differential operator so that you do not have to worry about converting operators for a two-particle system to differential operator language. We define the center of mass and relative coordinates by

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \qquad x = x_1 - x_2$$

The wave function can be considered as a function of X and x rather than x_1, x_2. Write down the time-independent Schrödinger equation for this problem, with H acting on $\psi(X, x)$. You can use the chain rule to evaluate the derivatives, e.g.,

$$\frac{\partial}{\partial x_1} \psi(X, x) = \frac{\partial X}{\partial x_1} \frac{\partial \psi}{\partial X} + \frac{\partial x}{\partial x_1} \frac{\partial \psi}{\partial x} = \frac{m_1}{m_1 + m_2} \frac{\partial \psi}{\partial X} + \frac{\partial \psi}{\partial x}$$

Complete the calculation expressing the equation in terms of functions of X, x and derivatives with respect to these. Notice that V is already a function of x.

Solution

Let $M = m_1 + m_2$. We already have

$$\frac{\partial}{\partial x_1} \psi(X, x) = \frac{m_1}{M} \frac{\partial \psi}{\partial X} + \frac{\partial \psi}{\partial x} = \left(\frac{m_1}{M} \frac{\partial}{\partial X} + \frac{\partial}{\partial x} \right) \psi$$

In a similar way

$$\frac{\partial}{\partial x_2} \psi(X, x) = \left(\frac{m_2}{M} \frac{\partial}{\partial X} - \frac{\partial}{\partial x} \right) \psi$$

Thus

$$\frac{\partial^2 \psi}{\partial x_1^2} = \left(\frac{m_1}{M}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}\right)\left(\frac{m_1}{M}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}\right)\psi$$

$$= \frac{m_1^2}{M^2}\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial x^2} + 2\frac{m_1}{M}\frac{\partial^2 \psi}{\partial X \partial x}$$

$$\frac{\partial^2 \psi}{\partial x_2^2} = \left(\frac{m_2}{M}\frac{\partial}{\partial X} - \frac{\partial}{\partial x}\right)\left(\frac{m_2}{M}\frac{\partial}{\partial X} - \frac{\partial}{\partial x}\right)\psi$$

$$= \frac{m_2^2}{M^2}\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial x^2} - 2\frac{m_2}{M}\frac{\partial^2 \psi}{\partial X \partial x}$$

The Hamiltonian is thus

$$H = -\frac{\hbar^2}{2}\left(\frac{1}{m_1}\frac{\partial^2 \psi}{\partial x_1^2} + \frac{1}{m_2}\frac{\partial^2 \psi}{\partial x_1^2}\right) + V(x)$$

$$= -\frac{\hbar^2}{2}\left(\frac{m_1}{M^2} + \frac{m_2}{M^2}\right)\frac{\partial^2 \psi}{\partial X^2} - \frac{\hbar^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{\partial^2 \psi}{\partial x^2} + V(x)$$

$$= -\frac{\hbar^2}{2M}\frac{\partial^2 \psi}{\partial X^2} + \left[-\frac{\hbar^2}{2\mu}\frac{\partial^2 \psi}{\partial x^2} + V(x)\right]$$

where

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

μ is the reduced mass. The cross-derivative terms cancel out. The final Hamiltonian shows that the center of mass motion is as a free particle of mass M, while the dynamics of the relative motion has the reduced mass and the potential.

Problem 4

In this problem, we consider the Hamiltonian for an isotropic three-dimensional harmonic oscillator given by

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + \frac{m\omega^2}{2}(x_1^2 + x_2^2 + x_3^2)$$

a) Calculate $[L_i, H]$.

b) From the commutator in part a, what is $\langle \partial L_i/\partial t \rangle$?

c) Write down the wave functions for the first excited state (there should be three such states) and identify the values of L_3 and L^2 for them.

Solution

a) We have already shown in problem 2 that $[L_i, x_1^2 + x_2^2 + x_3^2] = [L_i, p_1^2 + p_2^2 + p_3^2] = 0$. Thus $[L_i, H] = 0$.

b) We also have

$$i\hbar\frac{\partial}{\partial t}\langle\alpha|L_i|\alpha\rangle = \langle\alpha|L_iH - HL_i|\alpha\rangle = \langle\alpha|[L_i, H]|\alpha\rangle = 0$$

c) The Hamiltonian consists of three harmonic oscillators corresponding to the three directions, of the same frequency and mass. Thus we can define

$$a_i = \sqrt{\frac{m\omega}{2\hbar}}x_i + i\frac{p_i}{\sqrt{2m\hbar\omega}}, \qquad a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x_i - i\frac{p_i}{\sqrt{2m\hbar\omega}}$$

$$x_i = \sqrt{\frac{\hbar}{2m\omega}}(a_i + a_i^\dagger), \qquad p_i = -i\sqrt{\frac{m\hbar\omega}{2}}(a_i - a_i^\dagger)$$

where the subscript $i = 1, 2, 3$. The first set of excited states are given by $a_i^\dagger |0\rangle$, there are three of them. Also

$$
\begin{aligned}
L_3 &= x_1 p_2 - x_2 p_1 = -i\sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\hbar\omega}{2}}\left[(a_1 + a_1^\dagger)(a_2 - a_2^\dagger) - (a_2 + a_2^\dagger)(a_1 - a_1^\dagger)\right] \\
&= -i\hbar(a_1^\dagger a_2 - a_2^\dagger a_1) \\
L_1 &= -i\hbar(a_2^\dagger a_3 - a_3^\dagger a_2), \qquad L_2 = -i\hbar(a_3^\dagger a_1 - a_1^\dagger a_3)
\end{aligned}
$$

These may be combined as $L_i = -i\hbar \sum_{j,k}\epsilon_{ijk}a_j^\dagger a_k$. Thus

$$
\begin{aligned}
L_i L_i &= (-i\hbar)^2 \sum \epsilon_{ijk}\epsilon_{ibc}a_j^\dagger a_k a_b^\dagger a_c = -\hbar^2 \sum_{j,k} a_j^\dagger a_k(a_j^\dagger a_k - a_k^\dagger a_j) \\
&= -\hbar^2 \sum \left[a_j^\dagger(a_j^\dagger a_k + \delta_{jk})a_k - a_j^\dagger(a_k^\dagger a_k + \delta_{kk})a_j\right] \\
&= \hbar^2 \left[2a_k^\dagger a_k - a_j^\dagger a_j^\dagger a_k a_k + a_j^\dagger a_k^\dagger a_k a_j\right]
\end{aligned}
$$

On a state with only one $a^\dagger$, the last two terms in L^2 give zero since there are two a's, using $a_k |0\rangle = 0$. Thus $L^2 a_i^\dagger |0\rangle = 2\hbar^2 a_i^\dagger |0\rangle$. This corresponds to $l = 1$. For the action of L_3, we find from the given expression,

$$
\begin{aligned}
L_3 a_3^\dagger |0\rangle &= -i\hbar(a_1^\dagger a_2 - a_2^\dagger a_1)a_3^\dagger |0\rangle = -i\hbar(a_1^\dagger a_3^\dagger a_2 |0\rangle - a_2^\dagger a_3^\dagger a_1 |0\rangle) = 0 \\
L_3 a_1^\dagger |0\rangle &= -i\hbar(a_1^\dagger a_2 - a_2^\dagger a_1)a_1^\dagger |0\rangle = i\hbar a_2^\dagger |0\rangle \\
L_3 a_2^\dagger |0\rangle &= -i\hbar(a_1^\dagger a_2 - a_2^\dagger a_1)a_2^\dagger |0\rangle = -i\hbar a_1^\dagger |0\rangle \\
L_3 (a_1^\dagger \pm i a_2^\dagger) |0\rangle &= \pm\hbar(a_1^\dagger \pm i a_2^\dagger) |0\rangle
\end{aligned}
$$

For $l = 1$, we should get 3 states with L_3 values equal to $\hbar, 0, -\hbar$. So the first excited states of the 3d oscillator can be identified as

$$|1, 0\rangle = a_3^\dagger |0\rangle, \qquad |1, 1\rangle = (a_1^\dagger + i a_2^\dagger) |0\rangle, \qquad |1, -1\rangle = (a_1^\dagger - i a_2^\dagger) |0\rangle$$