PHY V2500: QUANTUM MECHANICS I

Problem Set 5

Due November 11, 2025

Problem 1

In class I derived the formulae for the action of $J_\pm$ on the angular momentum eigenstates $|j, m\rangle$,

$$\begin{aligned} J_+ |j, m\rangle &= \hbar\sqrt{j(j+1) - m(m+1)}\ |j, m+1\rangle \\ J_- |j, m\rangle &= \hbar\sqrt{j(j+1) - m(m-1)}\ |j, m-1\rangle \end{aligned}$$

where $J_\pm = J_1 \pm iJ_2$. You will find these formulae in my notes as well. Recall also that

$$J_3 |j, m\rangle = m\hbar\ |j, m\rangle$$

a) Use these formulae to calculate J_1, J_2, J_3 as matrices for $j = 1$. (I worked out the case of $j = \frac{1}{2}$ in class obtaining the Pauli matrices.) In the present case, you should get 3×3 matrices.

b) Use them to calculate the commutators $[J_i, J_j]$ for all i, j. Use matrix multiplication, not operator identities.

Problem 2

The orbital angular momentum operator is defined as

$$\vec{L} = \vec{x} \times \vec{p}, \implies L_1 = x_2 p_3 - x_3 p_2,\ \ L_2 = x_3 p_1 - x_1 p_3,\ \ L_3 = x_1 p_2 - x_2 p_1$$

Using this definition and the Heisenberg algebra, calculate the commutators $[L_i, x_j]$, $[L_i, p_j]$, for all $i, j = 1, 2, 3$. Calculate also $[L_i, x_1^2 + x_2^2 + x_3^2]$.

Problem 3

We consider a two-particle system in one dimension with the Hamiltonian

$$H = -\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + V(x_1 - x_2)$$

x_1 and x_2 refer to the particles labeled 1 and 2 and likewise for the masses. I have written this as a differential operator so that you do not have to worry about converting operators for a two-particle system to differential operator language. We define the center of mass and relative coordinates by

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \qquad x = x_1 - x_2$$

The wave function can be considered as a function of X and x rather than x_1, x_2. Write down the time-independent Schrödinger equation for this problem, with H acting on $\psi(X,x)$. You can use the chain rule to evaluate the derivatives, e.g.,

$$\frac{\partial}{\partial x_1}\psi(X,x) = \frac{\partial X}{\partial x_1}\frac{\partial \psi}{\partial X} + \frac{\partial x}{\partial x_1}\frac{\partial \psi}{\partial x} = \frac{m_1}{m_1+m_2}\frac{\partial \psi}{\partial X} + \frac{\partial \psi}{\partial x}$$

Complete the calculation expressing the equation in terms of functions of X, $x$and derivatives with respect to these. Notice that V is already a function of x.

Problem 4

In this problem, we consider the Hamiltonian for an isotropic three-dimensional harmonic oscillator given by

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + \frac{m\omega^2}{2}(x_1^2 + x_2^2 + x_3^2)$$

a) Calculate $[L_i, H]$.

b) From the commutator in part a, what is $\langle \partial L_i/\partial t \rangle$?

c) Write down the wave functions for the first excited state (there should be three such states) and identify the values of L_3 and L^2 for them.