PHY V2500: QUANTUM MECHANICS I

Problem Set 6

Due November 25, 2025

Problem 1

The Hamiltonian for a rotating body is given by

$$H = \frac{1}{2\,I} \sum_j L_j L_j + \mu B \, L_3$$

where μ, B are constants; I is the moment of inertia, taken to be another constant.

a) Obtain the eigenvalues and eigenstates of this Hamiltonian using angular momentum theory.

b) What are the degeneracies for the energy eigenstates?

c) Obtain the time-evolution of $\langle L_i \rangle$ (for all i) using

$$i\hbar \frac{\partial \langle L_i \rangle}{\partial t} = \langle L_i H - H L_i \rangle$$

Solution

a) Since L^2 and L_3 can be simultaneously diagonalized, the standard eigenstates $|l, m\rangle$ will be eigenstates of the given Hamiltonian; i.e,

$$H\,|l, m\rangle = E_{l,m}\;|l, m\rangle\,, \qquad E_{l,m} = \left[\frac{\hbar^2}{2I} l(l+1) + \mu B\, m\hbar\right]$$

where l is a positive integer $0, 1, 2$,etc., and $m = -l, -l+1, \cdots, l$ for a given l.

b) For general values of I and μB, there is no degeneracy, there is only one state for a given energy.

c) L_i commutes with L^2, so we find

$$[L_i, L_3] = i\hbar\, \epsilon_{i3j}\, L_j$$

Thus

$$\begin{aligned}
\frac{\partial \langle L_1 \rangle}{\partial t} &= -\mu B\, \langle L_2 \rangle \\
\frac{\partial \langle L_2 \rangle}{\partial t} &= \mu B\, \langle L_1 \rangle \\
\frac{\partial \langle L_3 \rangle}{\partial t} &= 0
\end{aligned}$$

Problem 2

In the $|nlm\rangle$ notation, the wave functions for the electron in the Hydrogen atom for $n = 2$

are given by

$$\begin{aligned}
\langle \vec{x}|210\rangle = \psi_{210} &= \frac{1}{\sqrt{32\pi}}\frac{r}{a^{\frac{5}{2}}}\, e^{-r/2a}\, \cos\theta \\
\langle \vec{x}|21\pm 1\rangle = \psi_{21\pm 1} &= \mp\frac{1}{\sqrt{64\pi}}\frac{r}{a^{\frac{5}{2}}}\, e^{-r/2a}\, \sin\theta\, e^{\pm i\varphi}
\end{aligned}$$

where $a = \hbar^2/me^2$ is the Bohr radius. For each of these wave functions calculate the expectation value of $\sin\theta$, $\sin^2\theta$ and r.

Solution

We will first write out the integrals we will need. For the angular part, these are integrals of $\sin\theta\cos^2\theta$, $\sin^2\theta\cos^2\theta$, $\sin^3\theta$, $\sin^4\theta$ with the integration measure $d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta$.

$$\begin{aligned}
\int d\Omega(\sin\theta\cos^2\theta) &= 2\pi\int_0^{\pi} d\theta(\sin\theta\cos\theta)^2 = 2\pi\int_0^{\pi} d\theta\frac{1}{4}(\sin 2\theta)^2 \\
&= 2\pi\int_0^{\pi} d\theta\frac{1}{4}\left[\frac{1-\cos 4\theta}{2}\right] = \frac{\pi^2}{4} \\
\int d\Omega(\sin^2\theta\cos^2\theta) &= 2\pi\int_{-1}^{1} dz z^2(1-z^2) \qquad \Big[z = \cos\theta \\
&= \frac{8\pi}{15} \\
\int d\Omega(\sin^3\theta) &= 2\pi\int_0^{\pi} d\theta(\sin\theta)^4 = 2\pi\frac{1}{4}\int_0^{\pi} d\theta(1-\cos 2\theta)^2 \\
&= \frac{\pi}{2}\int_0^{\pi} d\theta\left[1 - 2\cos 2\theta + \left(\frac{1+\cos 4\theta}{2}\right)\right] \\
&= \frac{3\pi^2}{4} \\
\int d\Omega(\sin^4\theta) &= 2\pi\int_0^{\pi} d\theta(\sin\theta)^5 = 2\pi\int_{-1}^{1} dz(1-z^2)^2 \\
&= \frac{32\pi}{15}
\end{aligned}$$

$$\begin{aligned}
\langle 210|\sin\theta\,|210\rangle &= \frac{1}{32\pi}\int_0^{\infty} r^2 dr\,\frac{r^2}{a^5}e^{-r/a}\int d\Omega\sin\theta\cos^2\theta \\
&= \frac{1}{32\pi}(4!)\frac{\pi^2}{4} = \frac{3\pi}{16}
\end{aligned}$$

$$\begin{aligned}
\langle 210|\sin^2\theta\,|210\rangle &= \frac{1}{32\pi}\int_0^{\infty} r^2 dr\,\frac{r^2}{a^5}e^{-r/a}\int d\Omega\sin^2\theta\cos^2\theta \\
&= \frac{1}{32\pi}(4!)\frac{8\pi}{15} = \frac{2}{5}
\end{aligned}$$

$$\begin{aligned}
\langle 21\pm 1|\sin\theta\,|21\pm 1\rangle &= \frac{1}{64\pi}\int_0^{\infty} r^2 dr\,\frac{r^2}{a^5}e^{-r/a}\int d\Omega\sin^3\theta \\
&= \frac{1}{64\pi}(4!)\frac{3\pi^2}{4} = \frac{9\pi}{32}
\end{aligned}$$

$$\begin{aligned}\langle 21\pm 1|\sin^2\theta\,|21\pm 1\rangle &= \frac{1}{64\pi}\int_0^\infty r^2 dr\,\frac{r^2}{a^5}e^{-r/a}\int d\Omega\sin^4\theta \\ &= \frac{1}{64\pi}(4!)\frac{32\pi}{15} = \frac{4}{5}\end{aligned}$$

For the expectation value of r, we find

$$\begin{aligned}\langle 210|\,r\,|210\rangle &= \frac{1}{32\pi}\int_0^\infty r^2 dr\,\frac{r^2}{a^5}e^{-r/a}\,r\int d\Omega\cos^2\theta \\ &= a\frac{1}{32\pi}5!\frac{4\pi}{3} = 5a\end{aligned}$$

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Problem 3

A nucleus which is rotating is described by the Hamiltonian

$$H = \frac{L_1^2 + L_2^2}{2I_1} + \frac{L_3^2}{2I_3}$$

where $\vec{L}$ is the orbital angular momentum and I_1, I_3 are two of the principal moments of inertia, with $I_1 \neq I_3$. (They are not operators.) Obtain the eigenstates and eigenvalues of this Hamiltonian.

Solution

This is similar to Problem 1, but unlike that case, here the first term in H has only $L_1^2 + L_2^2$. So we rewrite it as

$$H = \frac{L_1^2 + L_2^2 + L_3^2}{2I_1} - \frac{L_3^2}{2I_1} + \frac{L_3^2}{2I_3} = \frac{L^2}{2I_1} + L_3^2\left(\frac{1}{2I_3} - \frac{1}{2I_1}\right)$$

The eigenstates are again $|l,m\rangle$, with the eigenvalue equation

$$H\,|l,m\rangle = E_{l,m}\,|l,m\rangle\,, \qquad E_{l,m} = \frac{l(l+1)\hbar^2}{2I_1} + m^2\hbar^2\left(\frac{1}{2I_3} - \frac{1}{2I_1}\right)$$

Problem 4

The time-derivative of the expectation value of the angular momentum is given by

$$i\hbar\frac{\partial}{\partial t}\langle L_i\rangle = \Big\langle L_i\,H - H\,L_i\Big\rangle$$

where the angular brackets indicate the expectation value for some state. Consider a particle with electric dipole moment $\vec{\mu}_1$ interacting with a fixed dipole of dipole moment $\vec{\mu}_2$. The Hamiltonian for dipole 1 is given by

$$H = \frac{p^2}{2M} - \frac{3\,\vec{x}\cdot\vec{\mu}_1\,\vec{x}\cdot\vec{\mu}_2}{r^5} + \frac{\vec{\mu}_1\cdot\vec{\mu}_2}{r^3}$$

Calculate the required commutator to obtain the rate of change of angular momentum. This is the quantum version of the usual torque equation in mechanics.

Solution

We know from previous problem sets that $[L_i, p^2] = 0$, $[L_i, r] = 0$. So the only term in H which can have a nonzero commutator with L_i is the term with

$$\vec{x}\cdot\vec{\mu}_1\vec{x}\cdot\vec{\mu}_2 = x_a\,x_b\,\mu_{1a}\mu_{2b}$$

We also have

$$[L_i, x_a] = i\hbar\epsilon_{iac}x_c$$

Thus

$$\begin{aligned}
[L_i, H] &= -\frac{3}{r^5}i\hbar\left[\epsilon_{iac}x_c x_b + \epsilon_{ibc}x_a x_c\right]\mu_{1a}\mu_{2b} \\
&= -\frac{3}{r^5}\left[(\vec{\mu}_1\times\vec{x})_i(\vec{\mu}_2\cdot\vec{x}) + (\vec{\mu}_1\cdot\vec{x})(\vec{\mu}_2\times\vec{x})_i\right]i\hbar
\end{aligned}$$

For the time-derivative of $\langle L_i\rangle$, we then find

$$\frac{\partial\langle L_i\rangle}{\partial t} = -\frac{3}{r^5}\left[(\vec{\mu}_1\times\vec{x})_i(\vec{\mu}_2\cdot\vec{x}) + (\vec{\mu}_1\cdot\vec{x})(\vec{\mu}_2\times\vec{x})_i\right]$$